MATH 101 Mathematics for Social Sciences I

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Outline

Lecture 9

Chapter 10 Limits and Continuity

- Sec. 10.2 Limits (continued)
- Sec. 10.3 Continuity

Infinite limits

If $f(x)$ increases without bound as x approaches a, we express this as

$$\lim_{x \to a} f(x) = +\infty = \infty.$$

If $f(x)$ decreases without bound as x approaches a, we express this as

$$\lim_{x \to a} f(x) = -\infty.$$

Caution

Note that the limits do not actually exist in this case! With this notation we specify how the function values $f(x)$ behave as x-values get closer to a.

Examples

1. Let $f(x) = \dfrac{1}{x^2}$.
   As $x \to 0$, the function values $1/x^2$ increase without bound. So,
   $$\lim_{x \to 0} \frac{1}{x^2} = +\infty.$$
2. Let $g(x) = \dfrac{1}{x}$.
   - If we approach 0 with positive x-values, then $1/x$ are positive and increase without bound. So, $\displaystyle\lim_{x \to 0^+} \frac{1}{x} = +\infty$.
   - If we approach 0 with negative x-values, then $1/x$ are negative and decrease without bound. So, $\displaystyle\lim_{x \to 0^-} \frac{1}{x} = -\infty$.
   - Since the one-sided limits are different, $\displaystyle\lim_{x \to 0} \frac{1}{x}$ does not exist.

Example

Find $\lim_{x \to 2} \frac{x+2}{x^2-4}$ if it exists.

Solution: We have

$$\lim_{x \to 2} \frac{x+2}{x^2-4} = \lim_{x \to 2} \frac{x+2}{(x+2)(x-2)} = \lim_{x \to 2} \frac{1}{x-2}.$$

In this case,

$$\lim_{x \to 2^+} \frac{1}{x-2} = +\infty \quad \text{and} \quad \lim_{x \to 2^-} \frac{1}{x-2} = -\infty.$$

So, $\lim_{x \to 2} \frac{x+2}{x^2-4}$ is neither ∞ nor $-\infty$.

Limits at infinity

If $f(x)$ gets closer to a finite number L as x increases without bound through positive values, we express this limit as

$$\lim_{x\to\infty} f(x) = L.$$

If $f(x)$ gets closer to a finite number M as x decreases without bound through negative values, we express this limit as

$$\lim_{x\to-\infty} f(x) = M.$$

Example

We have

$$\lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0$$

since the values of $1/x$ gets closer to 0 as we take x larger positive or negative without bound.

Remarks

1. If $p > 0$, then $\lim_{x \to \infty} \frac{1}{x^p} = 0$.

   If $1/x^p$ is also defined for $x < 0$, then $\lim_{x \to -\infty} \frac{1}{x^p} = 0$.

2. If $f(x)$ is a rational function with the leading terms $a_n x^n$ and $b_m x^m$ in the numerator and denominator, then

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}.$$

Examples

1. $\lim_{x\to\infty} \frac{1}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1}{x^{1/3}} = 0$
2. $\lim_{x\to\infty} \frac{x^2-1}{7-2x+8x^2} = \lim_{x\to\infty} \frac{x^2}{8x^2} = \lim_{x\to\infty} \frac{1}{8} = \frac{1}{8}$
3. $$\begin{aligned}\lim_{x\to-\infty} \frac{x}{(3x-1)^2} &= \lim_{x\to-\infty} \frac{x}{9x^2-6x+1} \\ &= \lim_{x\to-\infty} \frac{x}{9x^2} = \lim_{x\to-\infty} \frac{1}{9x} = \frac{1}{9} \lim_{x\to-\infty} \frac{1}{x} = 0\end{aligned}$$
4. $\lim_{x\to\infty} \frac{x^5-x^4}{x^4-x^3+2} = \lim_{x\to\infty} \frac{x^5}{x^4} = \lim_{x\to\infty} x = \infty$

Definition

A function f is **continuous** at a if and only if the following hold:

1. $f(a)$ exists
2. $\lim_{x \to a} f(x)$ exists
3. $f(a) = \lim_{x \to a} f(x)$

If f is not continuous at a, then we say that f is **discontinuous** at a, and a is a **point of discontinuity** of f.

We say that a function is **continuous on an interval** if it is continuous at each point in that interval.

Example

Show that $f(x) = x^2 - 3$ is continuous at -4.

Solution:

1. $f(-4) = (-4)^2 - 3 = 13$ exists.
2. $\lim_{x \to -4} f(x) = \lim_{x \to -4} (x^2 - 3) = (-4)^2 - 3 = 13$ exists.
3. $f(-4) = \lim_{x \to -4} f(x)$.

Therefore, f is continuous at -4.

Remark

A polynomial function $p(x)$ is continuous on its natural domain $(-\infty, \infty)$ since $\lim_{x \to a} p(x) = p(a)$ at every point a.

Possible points of discontinuity for a function f

1. f is not defined at a.
2. f has no limit as $x \to a$.
3. f is defined at a and has a limit as $x \to a$, but $f(a) \neq \lim_{x \to a} f(x)$.

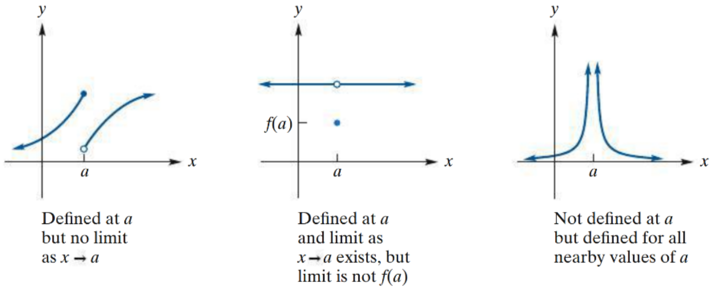

FIGURE 10.27 Discontinuities at a.

Remark

A rational function is discontinuous at points where the denominator is 0 and is continuous otherwise. So, a rational function is continuous on its domain.

Example

For each of the following functions, find all points of discontinuity.

a. $f(x) = \dfrac{x^2 - 3}{x^2 + 2x - 8}$

b. $g(x) = \dfrac{x + 4}{x^2 + 4}$

Solution:

a. We have the denominator equal to zero when

$$x^2 + 2x - 8 = 0 \Rightarrow (x + 4)(x - 2) = 0 \Rightarrow x = -4,\ x = 2.$$

So, f is discontinuous at -4 and 2.

b. Since $x^2 + 4$ is never zero, g has no discontinuity.

Example

Find all points where the following function is continuous.

$$f(x) = \begin{cases} x+6, & x \geq 3 \\ x^2, & x < 3 \end{cases}$$

Solution: Since $x+6$ and x^2 are polynomials they are continuous, so f is continuous for $x \neq 3$, but we need to analyse continuity at $x = 3$ separately.

1. $f(3) = 3 + 6 = 9$ exists.
2. Since $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} (x+6) = 9$ and $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} x^2 = 9$, the limit $\lim\limits_{x \to 3} f(x) = 9$ also exists.
3. We have $f(3) = \lim_{x \to 3} f(x) = 9$.

Therefore, f is also continuous at 3. So, f is continuous everywhere.