MATH 101 Mathematics for Social Sciences I

Can Deha Karıksız

Özyeğin University

Fall 2020/21

02.11.2020

Outline

Lecture 8

Chapter 4 Exponential and Logarithmic Functions

- Sec. 4.3 Properties of logarithms
- Sec. 4.4 Logarithmic and exponential equations

Chapter 10 Limits and Continuity

- Sec. 10.1 Limits

Rules for logarithms

Let m, n, r be real numbers and a, b be positive. Then,

1. $\log_b(mn) = \log_b m + \log_b n$
2. $\log_b \frac{m}{n} = \log_b m - \log_b n$
3. $\log_b m^r = r \log_b m$
4. $\log_b \frac{1}{m} = -\log_b m$
5. $\log_b 1 = 0$
6. $\log_b b = 1$
7. $\log_b m = \dfrac{\log_a m}{\log_a b}$

Example

Find $\log_7 \sqrt[9]{7^8}$.

Solution: $\log_7 \sqrt[9]{7^8} = \log_7 (7^8)^{1/9} = \log_7 7^{8/9} = \dfrac{8}{9}$.

Example

Write $\ln \sqrt[3]{\frac{x^5(x-2)^8}{x-3}}$ in terms of simpler logarithms.

Solution:

$$\begin{aligned}
\ln \sqrt[3]{\frac{x^5(x-2)^8}{x-3}} &= \ln \left(\frac{x^5(x-2)^8}{x-3}\right)^{1/3} \\
&= \frac{1}{3} \ln \frac{x^5(x-2)^8}{x-3} \\
&= \frac{1}{3}\left(\ln x^5 + \ln(x-2)^8 - \ln(x-3)\right) \\
&= \frac{5}{3} \ln x + \frac{8}{3} \ln(x-2) - \frac{1}{3} \ln(x-3)
\end{aligned}$$

Example

Express $\log x$ in terms of natural logarithms.

Solution: We apply the change of base formula $\log_b m = \dfrac{\log_a m}{\log_a b}$ with $b = 10$, $m = x$, and $a = e$ to get

$$\log x = \log_{10} x = \frac{\log_e x}{\log_e 10} = \frac{\ln x}{\ln 10}.$$

Example

Express $\log \frac{1}{x^2}$ in terms of $\log x$.

Solution: We have $\log \frac{1}{x^2} = \log x^{-2} = -2 \log x$ true for $x > 0$, but we can also take $x < 0$ since

$$\log \frac{1}{x^2} = \log \frac{1}{|x|^2} = \log |x|^{-2} = -2 \log |x|, \quad x \neq 0.$$

A **logarithmic equation** is an equation that involves the logarithm of an expression containing an unknown. Similarly, an **exponential equation** has the unknown appearing in an exponent. It is useful to remember that since the logarithm functions are one-to-one, we have

$$\log_b m = \log_b n \quad \Rightarrow \quad m = n.$$

Example

Solve $(25)^{x+2} = 5^{3x-4}$.

Solution: Since $25 = 5^2$, we have

$$(25)^{x+2} = 5^{3x-4} \Rightarrow (5^2)^{x+2} = 5^{3x-4} \Rightarrow 5^{2x+4} = 5^{3x-4}.$$

Taking any logarithm of both sides, we get

$$2x + 4 = 3x - 4 \Rightarrow x = 8.$$

Example

Solve $\log_2 x = 5 - \log_2(x+4)$.

Solution: Bringing the logarithms together, we have

$\log_2 x = 5 - \log_2(x+4) \Rightarrow \log_2 x + \log_2(x+4) = 5 \Rightarrow \log_2(x(x+4)) = 5.$

Writing the corresponding exponential form, we get

$$x(x+4) = 2^5$$
$$x^2 + 4x - 32 = 0$$
$$(x-4)(x+8) = 0$$
$$x = 4,\ x = -8$$

However, we must have $x > 0$ in the original equation, so the only possible solution is $x = 4$.

Definition

The **limit** of $f(x)$ as x approaches a is the number L, written

$$\lim_{x \to a} f(x) = L$$

if we can make the values $f(x)$ as close as we like to L by taking x sufficiently close to, but different from a. If there is no such number, we say that the limit of $f(x)$ as x approaches a does **not** exist.

Properties of Limits

Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, c be a constant, and n be a positive integer.

1. $\lim\limits_{x\to a} c = c$
2. $\lim\limits_{x\to a} x^n = a^n$
3. $\lim\limits_{x\to a}(f(x) \pm g(x)) = L \pm M$
4. $\lim\limits_{x\to a}(f(x).g(x)) = L.M$
5. $\lim\limits_{x\to a}(cf(x)) = cL$
6. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M},\ \ M \neq 0$
7. $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$

Examples

1. $\lim\limits_{x\to -1}(x^3 - x + 1) = \lim\limits_{x\to -1} x^3 - \lim\limits_{x\to -1} x + \lim\limits_{x\to -1} 1 = (-1)^3 - (-1) + 1 = 1$
2. $\lim\limits_{x\to 1} \dfrac{2x^2 + x - 3}{x^3 + 4} = \dfrac{\lim_{x\to 1}(2x^2 + x - 3)}{\lim_{x\to 1}(x^3 + 4)} = \dfrac{2 + 1 - 3}{1 + 4} = \dfrac{0}{5} = 0$
3. $\lim\limits_{x\to 3} \sqrt[3]{x^2 + 7} = \sqrt[3]{\lim\limits_{x\to 3}(x^2 + 7)} = \sqrt[3]{3^2 + 7} = \sqrt[3]{16} = 2\sqrt[3]{2}$

Remarks

1. If $f(x) = c_n x^n + \cdots + c_1 x + c_0$ is a polynomial function, then

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} (c_n x^n + \cdots + c_1 x + c_0) \\ &= c_n \lim_{x \to a} x^n + \cdots + c_1 \lim_{x \to a} x + \lim_{x \to a} c_0 \\ &= c_n a^n + \cdots + c_1 a + c_0 = f(a) \end{aligned}$$

2. If f and g are two functions for which $f(x) = g(x)$ for all $x \neq a$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

Example

Find $\lim_{x \to 1} \dfrac{x^3 - 1}{x - 1}$.

Solution: We have

$$\frac{x^3 - 1}{x - 1} = \frac{(x-1)(x^2+x+1)}{(x-1)} = x^2 + x + 1 \quad \text{for } x \neq 1.$$

Therefore,

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} (x^2 + x + 1) = (1)^2 + (1) + 1 = 3.$$

A special limit

$$\lim_{x \to 0} (1 + x)^{1/x} = e$$

One-sided limits

If the function values $f(x)$ get closer to a value L as x approaches a **from the right**, then we have

$$\lim_{x\to a^+} f(x) = L.$$

Similarly, if the function values $f(x)$ get closer to a value M as x approaches a **from the left**, then we have

$$\lim_{x\to a^-} f(x) = M.$$

Example

Let $f(x) = \sqrt{x-3}$. Then, since $f(x)$ is defined only when $x \geq 3$, we have

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \sqrt{x-3} = 0.$$

However, $\lim_{x\to 3^-} f(x)$ does not exist since f is not defined for $x < 3$.

Theorem

The limit $\lim_{x\to a} f(x)$ *exists if and only if both one-sided limits exist and*

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x).$$

Example

Let $f(x) = \begin{cases} x-1, & x<0, \\ 0, & x=0, \\ x+1, & x>0. \end{cases}$ Find $\lim_{x\to 0} f(x)$ if it exists.

Solution: Since the function is defined differently as we approach 0 from the left and the right, we can look at the one-sided limits.

We have, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x-1) = 0-1 = -1.$

On the other hand, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x+1) = 0+1 = 1.$

Since the one-sided limits are not equal, $\lim_{x\to 0} f(x)$ does not exist.