MATH 101 Mathematics for Social Sciences I

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Outline

Lecture 7

Chapter 4 Exponential and Logarithmic Functions

- Sec. 4.1 Exponential functions
- Sec. 4.2 Logarithmic functions

Definition

The function defined by

$$f(x) = b^x$$

where $b > 0$, $b \neq 1$, and the exponent is any real number, is called an **exponential function** with base b.

Rules for exponents

If x and y are real numbers and b and c are positive, we have

1. $b^x b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(bc)^x = b^x c^x$
5. $\left(\dfrac{b}{c}\right)^x = \dfrac{b^x}{c^x}$
6. $b^1 = b$
7. $b^0 = 1$
8. $b^{-x} = \dfrac{1}{b^x}$

Example

The number of bacteria present in a culture after t minutes is given by

$$N(t) = 300\left(\frac{4}{3}\right)^t.$$

a. How many bacteria are present initially?
b. Approximately how many bacteria are present after 3 minutes?

Solution:

a. We need to find $N(t)$ when $t = 0$. So, we have

$$N(0) = 300\left(\frac{4}{3}\right)^0 = 300(1) = 300.$$

b. $N(3) = 300\left(\frac{4}{3}\right)^3 = 300\left(\frac{4^3}{3^3}\right) = 300\left(\frac{64}{27}\right) = \frac{6400}{9} \approx 711.$

Properties of the exponential function $f(x) = b^x$

1. The domain of f is $(-\infty, \infty)$ and the range of f is $(0, \infty)$.
2. The graph of $y = b^x$ has y-intercept $(0, 1)$. There is no x-intercept.
3. If $b > 1$, the graph rises from left to right. If $0 < b < 1$, the graph falls from left to right.
4. If $b > 1$, the graph approaches the x-axis as x becomes more negative. If $0 < b < 1$, the graph approaches the x-axis as x becomes more positive.

Compound interest

The compound amount S of a principal P at the end of n interest periods at the periodic rate of r is given by

$$S = P(1 + r)^n.$$

Example

Suppose 100,000 Liras is invested for 5 years at an annual rate of 12% compounded monthly. Find the compounded amount at the end of investment.

Solution: We have the principal $P = 100,000$ liras, the monthly rate $r = 0.12/12 = 0.01$, and the interest period $n = 5 \times 12 = 60$ months. So,

$$S = P(1 + r)^n = 100,000(1 + 0.01)^{60} \approx 181,669.67$$

The number e

The smallest real number that is greater than all of the numbers $\left(1+\frac{1}{n}\right)^n$ is called **Euler's number** and it is denoted by e. The number e is irrational, so it has a nonrepeating decimal expansion, and $e \approx 2.71828$.

The exponential function $f(x) = e^x$ with base e is called the **natural exponential function**.

Example

The projected population P of a city is given by $P = 100,000e^{0.05t}$, where t is the number of years after 2000. Predict the population for the year 2020.

Solution: Since $t = 2020 - 2000 = 20$, we have

$$P = 100,000e^{0.05(20)} = 100,000e^{1} = 100,000e \approx 271,828$$

If $f(x) = b^x$ is an exponential function base b ($b > 0$, $b \neq 1$), then it passes the horizontal line test, so it has an inverse function $f^{-1}(x)$ called the **logarithm function base** b. It is denoted by $\log_b x$, and we have

$$y = \log_b x \quad \Leftrightarrow \quad b^y = x.$$

Since the exponential and the logarithm functions with the same base are inverses of each other, we have

$$\log_b b^x = x \quad \text{and} \quad b^{\log_b x} = x.$$

Recall that the domain of an exponential function is $(-\infty, \infty)$ and its range is $(0, \infty)$. Therefore, the domain of a logarithm function is $(0, \infty)$ and the range is $(-\infty, \infty)$.

Example

Find $\log_2 8$.

Solution: We can approach the solution in two similar ways.

1. $y = \log_2 8$ means $2^y = 8$. Since $8 = 2^3$ we have $y = 3$.
2. Since $\log_2 2^x = x$, we have $\log_2 8 = \log_2 2^3 = 3$.

Example

Find $\log_{64} 8$.

Solution: Since $8 = \sqrt{64} = 64^{1/2}$, we have $\log_{64} 8 = \log_{64} 64^{1/2} = \frac{1}{2}$.

Example

Find $\log_2 \frac{1}{16}$.

Solution: Since $\frac{1}{16} = \frac{1}{2^4} = 2^{-4}$, we have $\log_2 \frac{1}{16} = \log_2 2^{-4} = -4$.

Graph of the logarithmic function $f(x) = \log_b x$

1. The graph of $y = \log_b x$ is the mirror image of $y = b^x$ in the line $y = x$.
2. The graph of $y = \log_b x$ has x-intercept $(1, 0)$. There is no y-intercept.
3. If $b > 1$, the graph rises from left to right. If $0 < b < 1$, the graph falls from left to right.
4. If $b > 1$, the graph gets closer to the y-axis and the function values decrease without bound as x gets closer to 0. If $0 < b < 1$, the graph gets closer to the y-axis and the function values increase without bound as x gets closer to 0.

Special logarithm bases

The logarithm to the base e is called the **natural logarithm**:

$$\ln x \quad \text{means} \quad \log_e x.$$

The logarithm to the base 10 is called the **common logarithm**, denoted by

$$\log x \quad \text{means} \quad \log_{10} x.$$

Examples

1. $\log 100 = log_{10}100 = log_{10}10^2 = 2.$
2. $\ln 1 = \log_e 1 = \log_e e^0 = 0.$
3. $\log 0.1 = \log_{10} 0.1 = \log_{10} 10^{-1} = -1.$
4. $\ln \frac{1}{e} = \ln e^{-1} = -1.$

Example

Solve $\ln(x+1) = 7$.

Solution: $\ln(x+1) = 7$ means $e^7 = x + 1$. Therefore, $x = e^7 - 1$.

Example

Solve $\log_x 49 = 2$.

Solution: $\log_x 49 = 2$ means $x^2 = 49$. This equation has two roots $x = 7$ and $x = -7$, but the base of a logarithm can only be positive, so the only possible solution is $x = 7$.

Example

Solve $e^{5x} = 4$.

Solution: $e^{5x} = 4$ in logarithmic form is $\ln 4 = 5x$. Therefore, $x = \dfrac{\ln 4}{5}$.

Radioactive decay

If N is the amount of a radioactive element at time t, then the element decays exponentially by the formula

$$N = N_0 e^{-\lambda t}$$

where the initial amount N_0 and the decay constant λ are positive constants.

Example

The **half-life** of a radioactive element is the interval of time required for one-half of a radioactive sample to decay. Find a formula for the half-life.

Solution: Let T denote the time required for one-half of the initial amount N_0 to decay. From the identity $N = N_0 e^{-\lambda t}$, we have

$$N_0 e^{-\lambda T} = \frac{N_0}{2} \Rightarrow e^{-\lambda T} = \frac{1}{2} \Rightarrow e^{\lambda T} = 2 \Rightarrow \lambda T = \ln 2 \Rightarrow T = \frac{\ln 2}{\lambda}$$