MATH 101 Mathematics for Social Sciences I

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Outline

Lecture 6

Chapter 3 Lines, Parabolas, and Systems

- Sec. 3.2 Applications and linear functions
- Sec. 3.3 Quadratic functions

Recall from before

A function f is a **linear function** if and only if $f(x) = ax + b$, where a and b are constants and $a \neq 0$.

The graph of $y = f(x) = ax + b$ is a straight line with slope a and y-intercept b.

Example

Suppose f is a linear function with slope 2 and $f(4) = 8$. Find $f(x)$.

Solution: Since f is linear, it has the form $f(x) = ax + b$.
The slope is 2, so $a = 2$, which implies $f(x) = 2x + b$.
Since $f(4) = 8$, we have $f(4) = 2(4) + b \Rightarrow 8 = 8 + b \Rightarrow b = 0$.
Hence, $f(x) = 2x$.

Example

If $y = f(x)$ is a linear function with $f(-2) = 6$ and $f(1) = -3$, find $f(x)$.

Solution: The graph of a linear function is a straight line, and since $f(-2) = 6$ and $f(1) = -3$, the line passes through the points $(x_1, y_1) = (-2, 6)$ and $(x_2, y_2) = (1, -3)$.

We have the slope $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-3 - 6}{1 - (-2)} = \dfrac{-9}{3} = -3$.

Using the point-slope form, we get

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 6 &= -3(x - (-2)) \\
y - 6 &= -3x - 6 \\
y = f(x) &= -3x.
\end{aligned}
$$

Recall from before

A function f is a **quadratic function** if and only if $f(x) = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

Graph of a quadratic function

1. The graph of a quadratic function $y = f(x) = ax^2 + bx + c$ is a **parabola**.
2. If $a > 0$ the parabola **opens upward**. If $a < 0$ it **opens downwards**.
3. Each parabola is symmetric about a vertical line, called the **axis of symmetry**.
4. The point where the axis cuts the parabola is called the **vertex**. The coordinates of the vertex are $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.
5. The y-intercept of the parabola is c.

Parabola: $y = f(x) = ax^2 + bx + c$

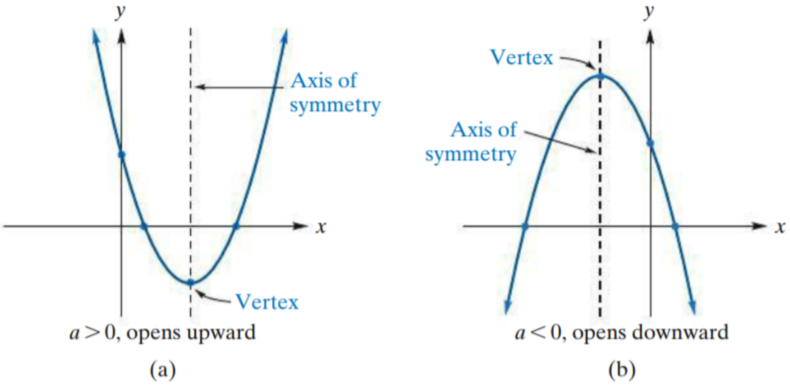


FIGURE 3.18 Parabolas.

We can quickly sketch the graph of a quadratic function by locating the vertex, the intercepts, and a few other points if necessary.

Example

Graph $y = f(x) = -x^2 - 4x + 12$.

Solution: f is a quadratic function with $a = -1$, $b = -4$ and $c = 12$, so its graph is a parabola. Since $a < 0$, the parabola opens downwards.
The coordinates of the vertex are given by

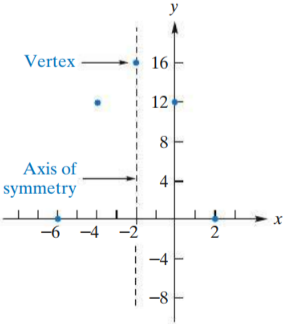
$$-\frac{b}{2a} = -\frac{-4}{2(-1)} = -2 \text{ and } f\left(-\frac{b}{2a}\right) = f(-2) = -(-2)^2 - 4(-2) + 12 = 16.$$

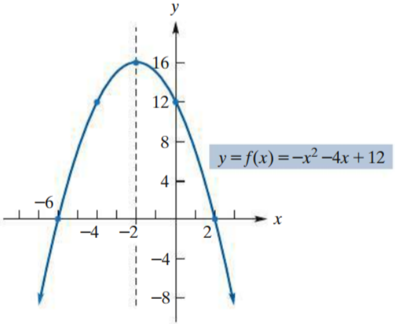
So, the vertex is at $(-2, 16)$.
Since $c = 12$, the y-intercept is at $(0, 12)$.

To find the x-intercepts we set $y = 0$ and solve for x:

$$0 = -x^2 - 4x + 12$$
$$0 = -(x+6)(x-2)$$
$$x = -6,\ x = 2$$

So, the x-intercepts are at $(-6, 0)$ and $(2, 0)$.





Example

Graph $y = g(x) = x^2 - 6x + 7$.

Solution: g is a quadratic function with $a = 1$, $b = -6$ and $c = 7$, so its graph is a parabola. Since $a > 0$, the parabola opens upwards.
The coordinates of the vertex are given by

$$-\frac{b}{2a} = -\frac{-6}{2(1)} = 3 \text{ and } f\left(-\frac{b}{2a}\right) = f(3) = (3)^2 - 6(3) + 7 = -2.$$

So, the vertex is at $(3, -2)$. Since $c = 7$, the y-intercept is at $(0, 7)$.
To find the x-intercepts, set $y = 0$ and solve for x:

$$0 = x^2 - 6x + 7$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{(-6) \pm \sqrt{(-6)^2 - 4(1)(7)}}{2(1)} = 3 \pm \sqrt{2}$$

So, the x-intercepts are at $(3 - \sqrt{2}, 0)$ and $(3 + \sqrt{2}, 0)$.

Example

Graph $y = f(x) = 2x^2 + 2x + 3$ and find the range of f.

Solution: f is a quadratic function with $a = 2$, $b = 2$ and $c = 3$, so its graph is a parabola. Since $a > 0$, the parabola opens upwards.
The coordinates of the vertex are given by

$$-\frac{b}{2a} = -\frac{2}{2(2)} = -\frac{1}{2} \text{ and } f\left(-\frac{b}{2a}\right) = f(-\frac{1}{2}) = 2(-\frac{1}{2})^2 + 2(-\frac{1}{2}) + 3 = \frac{5}{2}.$$

So, the vertex is at $(-\frac{1}{2}, \frac{5}{2})$. Since $c = 3$, the y-intercept is at $(0, 3)$.
Since the parabola is opening upwards and its vertex is above the x-axis, there are no x-intercepts. We can consider additional points on the graph to have a more accurate sketch.
From the graph we see that the set of all possible y-values on the graph is $y \geq 5/2$, so the range of f is $[5/2, \infty)$.

Example

The demand function for a product is $p = 1000 - 2q$, where p is the price per unit when q units are demanded by consumers. Find the quantity produced that will maximize the total revenue, and determine this revenue.

Solution: Using the relation

$$\text{total revenue} = (\text{price})(\text{quantity}),$$

we get

$$r = pq = (1000 - 2q)q = 1000q - 2q^2.$$

r is now a quadratic function of q with $a = -2$, $b = 1000$ and $c = 0$. Since $a < 0$ the parabola opens downwards, therefore the maximum is attained at the vertex. We have,

$$-\frac{b}{2a} = -\frac{1000}{2(-2)} = 250 \text{ and } r(250) = 1000(250) - 2(250)^2 = 125,000.$$

So the maximum possible revenue is 125,000 for the quantity of 250 units.