MATH 101 Mathematics for Social Sciences I

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Outline

Lecture 2

- Chapter 2 Functions and Graphs
  - Sec. 2.3 Combinations of functions
  - Sec. 2.4 Inverse functions

Definition

For any functions $f, g : X \to (-\infty, \infty)$, we define the **sum**, **difference**, **product** and **quotient** of f and g as follows:

$$
\begin{aligned}
(f+g)(x) &= f(x) + g(x) \\
(f-g)(x) &= f(x) - g(x) \\
(fg)(x) &= f(x)\,g(x) \\
\frac{f}{g}(x) &= \frac{f(x)}{g(x)}, \quad g(x) \neq 0
\end{aligned}
$$

For each of these new functions, the domain is the set of all x that belong to both the domain of f and the domain of g, with the domain of the quotient also excluding any value of x for which $g(x) = 0$.

Example

If $f(x) = 3x - 1$ and $g(x) = x^2 + 3x$, find

a. $(f+g)(x)$
b. $(f-g)(x)$
c. $(fg)(x)$
d. $\frac{f}{g}(x)$
e. $((1/2)f)(x)$

Solution:

a. $(f+g)(x) = f(x) + g(x) = (3x-1) + (x^2+3x) = x^2 + 6x - 1$
b. $(f-g)(x) = f(x) - g(x) = (3x-1) - (x^2+3x) = -1 - x^2$
c. $(fg)(x) = f(x)g(x) = (3x-1)(x^2+3x) = 3x^3 + 8x^2 - 3x$
d. $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{3x-1}{x^2+3x}$
e. $((1/2)f)(x) = (1/2)f(x) = (1/2)(3x-1) = \frac{3}{2}x - \frac{1}{2}$

Composition

For functions $g : X \to Y$ and $f : Y \to Z$, the **composite** of f with g is the function $f \circ g : X \to Z$ defined by

$$(f \circ g)(x) = f(g(x))$$

where the domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

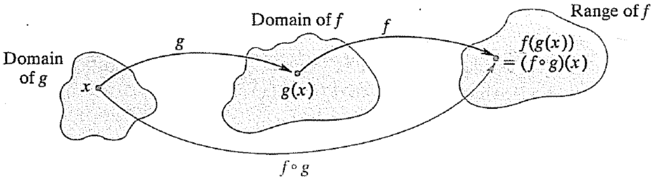

FIGURE 2.2 Composite of f with g.

Example

Let $f(x) = \sqrt{x}$ and $g(x) = x + 1$. Find

a. $(f \circ g)(x)$
b. $(g \circ f)(x)$

Solution:

a.

$$(f \circ g)(x) = f(g(x)) = f(x+1) = \sqrt{x+1}$$

Note that although $g(x) = x + 1$ can take any real number as an input, since $f(x) = \sqrt{x}$ can only take nonnegative values as input, the domain of $f \circ g$ is the interval $[-1, \infty)$.

b.

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x} + 1$$

The domain of $g \circ f$ is the interval $[0, \infty)$.

Remark: Generally, $f \circ g$ and $g \circ f$ are different functions!

Example

Express $h(x) = (2x - 1)^3$ as a composite.

Solution: To find $h(x)$, we need to find $2x - 1$ first and then cube the result. If we set $g(x) = 2x - 1$ and $f(x) = x^3$, then

$$h(x) = (2x-1)^3 = (g(x))^3 = f(g(x)) = (f \circ g)(x)$$

which gives h as a composite of two functions.

The identity function

The function I defined by $I(x) = x$ is called the **identity function**.

Note that for any function f, we have

$$\boxed{f \circ I = f = I \circ f}$$

Definition

Given a function $f : X \to Y$, if there exists a function f^{-1} satisfying

$$f \circ f^{-1} = I = f^{-1} \circ f$$

where I is the identity function, then f is called **invertible** and f^{-1} is called the **inverse** of f.

Remarks:

1. If a function f is invertible, then it has a unique inverse f^{-1}.
2. The equations above are equivalent to

$$f^{-1}(f(x)) = x \quad \text{for all } x \text{ in the domain of } f,$$
$$f(f^{-1}(y)) = y \quad \text{for all } y \text{ in the range of } f.$$

Not all functions are invertible! We need an additional condition on f for its inverse to exist.

Definition

A function f is called **one-to-one** if

$$f(x_1) = f(x_2) \quad \text{implies} \quad x_1 = x_2$$

for all x_1 and x_2 in the domain of f.

Another way to state the condition for f being one-to-one is

$$x_1 \neq x_2 \quad \text{implies} \quad f(x_1) \neq f(x_2).$$

This suggests that a one-to-one function should send distinct inputs to distinct outputs.

Theorem

A function f has an inverse f^{-1} if and only if it is one-to-one.

Example

Show that a linear function $f(x) = ax + b$, $a \neq 0$ is one-to-one, and find its inverse.

Solution: Assume $f(x_1) = f(x_2)$. Then, $ax_1 + b = ax_2 + b$. Subtracting b from both sides we get $ax_1 = ax_2$. Since $a \neq 0$, we can divide both sides by a to get $x_1 = x_2$. This shows that $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all x_1 and x_2. By definition, f is one-to-one.

To find the inverse of f, we need to recall that $f(f^{-1}(x)) = x$ for all x in the range of f. Applying f on the left hand side, we get $af^{-1}(x) + b = x$.

Solving for $f^{-1}(x)$ we obtain $f^{-1}(x) = \dfrac{x-b}{a} = \dfrac{1}{a}x - \dfrac{b}{a}$.

Note that the equation $f^{-1}(f(x)) = x$ is also satisfied, and $f^{-1}(x)$ is a linear function itself.

Example

a. Show that the function $f(x) = x^2$ is not invertible with its natural domain $(-\infty, \infty)$.
b. Show that $f(x) = x^2$ becomes invertible if its domain is restricted to the interval $[0, \infty)$.

Solution:

a. The condition for being one-to-one is not satisfied in this case. For example, if we take $x_1 = 1$ and $x_2 = -1$, then

$$f(x_1) = (1)^2 = 1 = (-1)^2 = f(x_2), \text{ but } x_1 = 1 \neq -1 = x_2.$$

b. If we restrict the domain to $[0, \infty)$, then taking x_1 and x_2 in $[0, \infty)$,

$$f(x_1) = f(x_2) \Rightarrow (x_1)^2 = (x_2)^2 \Rightarrow \sqrt{(x_1)^2} = \sqrt{(x_2)^2} \Rightarrow x_1 = x_2.$$

Example

Find $f^{-1}(x)$ if $f(x) = (x-1)^2$ for $x \geq 1$.

Solution: Note that f is one-to-one in the restricted domain $[1, \infty)$. An alternative way of finding the inverse of f is to solve the equation $y = f(x)$ for x in terms of y (if possible) and then switching y to x. Let $y = (x-1)^2$ for $x \geq 1$. Then $\sqrt{y} = x - 1$ and hence $x = \sqrt{y} + 1$. Switching y to x, we get $f^{-1}(x) = \sqrt{x} + 1$.

Some identities for inverses

1. If f and g are one-to-one, then $f \circ g$ is also one-to-one and
$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}.$$
2. If f is one-to-one, then $(f^{-1})^{-1} = f$.