MATH 101 Mathematics for Social Sciences I

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Outline

Lecture 10

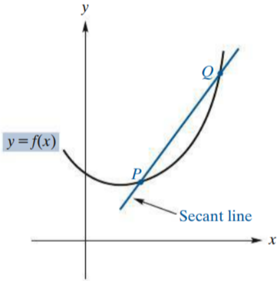
Chapter 11 Differentiation

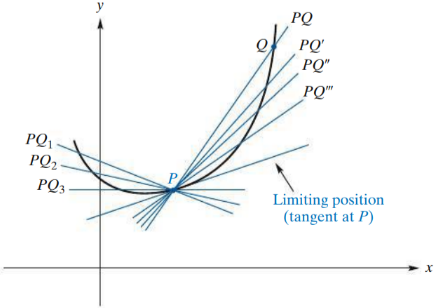
- Sec. 11.1 The derivative

Tangent lines to a curve

Consider a curve given by the graph of a function $y = f(x)$. A **secant line** is a line that intersects a curve at two or more points.

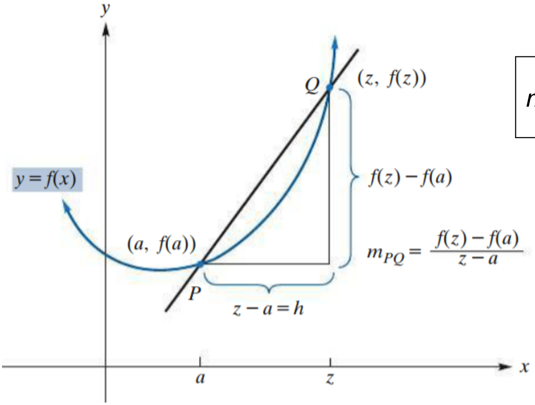
Given the secant lines PQ, if we fix the point P on the curve and can take the points Q closer to P from both directions, then the line on the limiting position is the **tangent line** to the curve at P.





Definition

The **slope of a curve** at a point P is the slope, if it exists, of the tangent line at P.



$$m_{\tan} = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example

Find the slope of the tangent line to the curve $y = f(x) = x^2$ at $(1, 1)$.

Solution: We use the limit formula with $f(x) = x^2$ and $a = 1$.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\
&= \lim_{h\to 0} \frac{(1+h)^2 - (1)^2}{h} \\
&= \lim_{h\to 0} \frac{1 + 2h + h^2 - 1}{h} \\
&= \lim_{h\to 0} \frac{h(2+h)}{h} = \lim_{h\to 0}(2+h) = 2
\end{aligned}$$

Therefore, the tangent line to $y = x^2$ at $(1, 1)$ has slope 2.

Definition

The **derivative** of a function f is the function denoted f' ("f prime") defined by

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

If $f'(a)$ can be found, then f is **differentiable** at a. The process of finding the derivative is called **differentiation**.

Notation

The derivative of $y = f(x)$ can be denoted in several different ways:

$$f'(x), \quad y', \quad \frac{dy}{dx}, \quad \frac{d}{dx}(f(x)), \quad D_x y, \quad D_x(f(x))$$

Example

If $f(x) = x^2$, find the derivative of f.

Solution: Applying the definition of the derivative we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$
$$= \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x$$

Note that when we are taking the limit as $h \to 0$, the values of x is not changing, so we treat the x term as constant.

Remark

$f'(a)$ is the slope of line tangent to the graph of $y = f(x)$ at $(a, f(a))$.

Notation: For $y = f(x)$, we have $f'(a) = y'(a) = \left.\dfrac{dy}{dx}\right|_{x=a}$.

Example

If $f(x) = 2x^2 + 2x + 3$, find an equation of the tangent line to the graph of f at $(1, 7)$.

Solution: To write the tangent line equation we can use the point-slope form. The slope of the tangent line is given by the derivative of f at $x = 1$, and $(1, 7)$ is a point on it.

To find the derivative, we have

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(2(x+h)^2+2(x+h)+3)-(2x^2+2x+3)}{h} \\
&= \lim_{h\to 0} \frac{4xh+2h^2+2h}{h} \\
&= \lim_{h\to 0} (4x+2h+2) = 4x+2
\end{aligned}
$$

Therefore, $f'(1) = 4(1)+2 = 6$ is the slope of the tangent line at $(1,7)$. Using the point-slope form, we get an equation of the tangent line as

$$
\begin{aligned}
y-7 &= 6(x-1) \\
y &= 6x+1
\end{aligned}
$$

Example

Find $\frac{d}{dx}(\sqrt{x})$.

Solution: Let us take $f(x) = \sqrt{x}$ and use the alternative limit definition of the derivative:

$$\begin{aligned}\frac{d}{dx}(\sqrt{x}) &= \lim_{z\to x}\frac{f(z)-f(x)}{z-x}\\ &= \lim_{z\to x}\frac{\sqrt{z}-\sqrt{x}}{z-x}\cdot\frac{\sqrt{z}+\sqrt{x}}{\sqrt{z}+\sqrt{x}}\\ &= \lim_{z\to x}\frac{z-x}{(z-x)(\sqrt{z}+\sqrt{x})}\\ &= \lim_{z\to x}\frac{1}{\sqrt{z}+\sqrt{x}} = \frac{1}{2\sqrt{x}}\end{aligned}$$

Note that although $\sqrt{x}$ is defined at $x = 0$, its derivative $1/(2\sqrt{x})$ does not exist when $x = 0$.

Theorem

If f is differentiable at a, then f is continuous at a.

Examples

1. Let $f(x) = x^2$. Its derivative $f'(x) = 2x$ is defined for all values of x, so f is differentiable everywhere, which implies that f is continuous everywhere.
2. The absolute value function $g(x) = |x|$ is continuous at $x = 0$, however it is not differentiable at $x = 0$ since it has no tangent line there.