ÖRNEKLER-MATRİSLER

1. $\mathbf{A}=\begin{bmatrix}3 & 3 & -1\\ -2 & -2 & 1\\ -4 & -5 & 2\end{bmatrix}$ ise $\mathbf{A}^{-1}=?$

Çözüm: Genişletilmiş matris ile bulunabilir.

$$[\mathbf{A}\,|\,\mathbf{I}]=\left[\begin{array}{ccc|ccc}3 & 3 & -1 & 1 & 0 & 0\\ -2 & -2 & 1 & 0 & 1 & 0\\ -4 & -5 & 2 & 0 & 0 & 1\end{array}\right]\Rightarrow R_2+R_1\rightarrow$$

$$=\left[\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 1 & 0\\ -2 & -2 & 1 & 0 & 1 & 0\\ -4 & -5 & 2 & 0 & 0 & 1\end{array}\right]\Rightarrow 2R_1+R_2$$

$$=\left[\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 2 & 3 & 0\\ -4 & -5 & 2 & 0 & 0 & 1\end{array}\right]\Rightarrow 4R_1+R_3$$

$$=\left[\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 2 & 3 & 0\\ 0 & -1 & 2 & 4 & 4 & 1\end{array}\right]\Rightarrow R_2\rightarrow R_3$$

$$=\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 2 & 4 & 4 & 1 \\ 0 & 0 & 1 & 2 & 3 & 0 \end{array}\right] \Rightarrow R_2 + R_1 \to$$

$$=\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 5 & 5 & 1 \\ 0 & -1 & 2 & 4 & 4 & 1 \\ 0 & 0 & 1 & 2 & 3 & 0 \end{array}\right] \Rightarrow -R_2 \to$$

$$=\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 5 & 5 & 1 \\ 0 & 1 & -2 & -4 & -4 & -1 \\ 0 & 0 & 1 & 2 & 3 & 0 \end{array}\right] \Rightarrow 2R_3 + R_2 \to$$

$$=\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 5 & 5 & 1 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 & 3 & 0 \end{array}\right] \Rightarrow -2R_3 + R_1 \to$$

$$=\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 & 3 & 0 \end{array}\right]$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$$

2. $\mathbf{A}=\begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ ise $\mathbf{A}^{-1}=?$

Çözüm: Elemanter işlemler ile bulunabilir.

$$\mathbf{A}=\begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \Rightarrow R_1+R_2 \rightarrow$$

$$\mathbf{A}=\begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

İkinci ve üçüncü satırlar eşit olduğundan,

$|\mathbf{A}|=0$ olup tekil matristir, tersi alınamaz.

3. $\mathbf{A}=\begin{bmatrix}-1 & 3 & 2\\ 0 & -2 & 1\\ 1 & 0 & -2\end{bmatrix}$ ise $ek(\mathbf{A})=?$

Çözüm: İlk aşamada tüm elemanlar için işaretli minörler bulunmalıdır:

$$\begin{bmatrix}\begin{vmatrix}-2 & 1\\ 0 & -2\end{vmatrix} & -\begin{vmatrix}0 & 1\\ 1 & -2\end{vmatrix} & \begin{vmatrix}0 & -2\\ 1 & 0\end{vmatrix}\\ -\begin{vmatrix}3 & 2\\ 0 & -2\end{vmatrix} & \begin{vmatrix}-1 & 2\\ 1 & -2\end{vmatrix} & -\begin{vmatrix}-1 & 3\\ 1 & 0\end{vmatrix}\\ \begin{vmatrix}3 & 2\\ -2 & 1\end{vmatrix} & -\begin{vmatrix}-1 & 2\\ 0 & 1\end{vmatrix} & \begin{vmatrix}-1 & 3\\ 0 & -2\end{vmatrix}\end{bmatrix}=\begin{bmatrix}4 & 1 & 2\\ 6 & 0 & 3\\ 7 & 1 & 2\end{bmatrix}$$

Bulunan matrisin transpozu ek matrise eşittir:

$$ek(\mathbf{A})=\begin{bmatrix}4 & 6 & 7\\ 1 & 0 & 1\\ 2 & 3 & 2\end{bmatrix}$$

4. $\mathbf{A} = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$ ise $\mathbf{A}^{-1} = ?$

Çözüm: Ek matris ve determinant kullanılarak bulunacak ise ilk aşamada determinant bulunur:

$$|\mathbf{A}| = -1\begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} + 0\begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix}$$

$$|\mathbf{A}| = -1(4) + 0 + 1(7)$$

$$|\mathbf{A}| = 3$$

İkinci aşamada ek matris bulunur:

$$ek(\mathbf{A}) = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

Ters matris:

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} ek(\mathbf{A})$$

$$\mathbf{A}^{-1} = \frac{1}{3}\begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{3}\begin{bmatrix} 4/3 & 2 & 7/3 \\ 1/3 & 0 & 1/3 \\ 2/3 & 1 & 2/3 \end{bmatrix}$$

5. $\mathbf{A}=\begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & -1 & 2 & 0 \\ -1 & 0 & 1 & -3 \\ 1 & 1 & 0 & -1 \end{bmatrix}$ ise $\mathbf{A}^{-1}$ matrisini bulmak

için genişletilmiş matrise uygulanan elemanter satır işlemlerini belirleyiniz.

$$[\mathbf{A}:\mathbf{I}]=\left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -3 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \to 1?$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -3 & -2 & -1 & 0 & 0 & 1 \end{array}\right] \to 2?$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 2 & 0 & 1 \end{array}\right] \to 3?$$

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \rightarrow 4?$$

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 3 & 2 & -3 & 0 & -1 \\ 0 & 1 & 0 & -4 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & -2 & -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 4 & 3 & -5 & 1 & -2 \end{array}\right] \rightarrow 5?$$

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 3 & 2 & -3 & 0 & -1 \\ 0 & 1 & 0 & -4 & -2 & 3 & 0 & 2 \\ 0 & 0 & 1 & -2 & -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3/4 & -5/4 & 1/4 & -2/4 \end{array}\right] \rightarrow 6?$$

$$\left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & -1/4 & 3/4 & -3/4 & 1/2 \\ 0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 3/4 & -5/4 & 1/4 & -2/4 \end{array}\right] \rightarrow$$

Çözüm: İlk aşama: R_1+R_3 ve $-R_1+R_3$

İkinci aşama: $-R_2$, R_2+R_1, R_2+R_3 ve $-2\,R_2+R_4$

Üçüncü aşama: $R_3 \leftrightarrow R_4$

Dördüncü aşama: $-R_3+R_1$, $2R_3+R_2$ ve $-2\ R_3+R_4$

Beşinci aşama: $(1/4)R_4$

Altıncı aşama: $-3R_4+R_1$, $4R_4+R_2$ ve $2R_4+R_3$

6. $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 6 & 1 \\ -2 & 1 & 10 \end{bmatrix}$ ise $\mathbf{A}$ matrisini bulunuz.

Çözüm: $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$ olduğu için $\mathbf{A}^{-1}$ matrisinin

ters matrisi bulunmalıdır.

$$\left[\mathbf{A}^{-1} : \mathbf{I}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 1 & 0 \\ -2 & 1 & 10 & 0 & 0 & 1 \end{array}\right] \rightarrow \begin{array}{c} -2R_1 + R_2 \\ 2R_1 + R_3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & 3 & -2 & 1 & 0 \\ 0 & 5 & 8 & 2 & 0 & 1 \end{array}\right] \rightarrow \begin{array}{c} (1/2)R_2 \\ -5R_2 + R_3 \\ -2R_2 + R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 3 & -1 & 0 \\ 0 & 1 & 3/2 & -1 & 1/2 & 0 \\ 0 & 0 & 1/2 & 7 & -5/2 & 1 \end{array}\right] \rightarrow \begin{array}{c} 2R_3 \\ (-3/2)R_3 + R_2 \\ 4R_3 + R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 59 & -21 & 8 \\ 0 & 1 & 0 & -22 & 8 & -3 \\ 0 & 0 & 1 & 14 & -5 & 2 \end{array}\right]$$

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A} = \begin{bmatrix} 59 & -21 & 8 \\ -22 & 8 & -3 \\ 14 & -5 & 2 \end{bmatrix}$$

7. $\mathbf{A}=\begin{bmatrix}1 & 6 & 3\\ 2 & 0 & 4\\ -5 & 2 & 3\end{bmatrix}$ matrisi için simetrik ve yarı simetrik matrisleri bulunuz.

Çözüm: İlk olarak transpoz matrisi bulunmalıdır.

$$\mathbf{A}^T=\begin{bmatrix}1 & 2 & -5\\ 6 & 0 & 2\\ 3 & 4 & 3\end{bmatrix}$$

Simetrik matris;

$$\mathbf{P}=\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^T\right)=\begin{bmatrix}1 & 4 & -1\\ 4 & 0 & 3\\ -1 & 3 & 3\end{bmatrix}$$

Yarı simetrik matris;

$$\mathbf{Q}=\frac{1}{2}\left(\mathbf{A}-\mathbf{A}^T\right)=\begin{bmatrix}0 & 2 & 4\\ -2 & 0 & 1\\ -4 & -1 & 0\end{bmatrix}$$

8. Aşağıda verilen matrislerin hangileri Echelon matristir.

a. $\mathbf{A}=\begin{bmatrix} 3 & 2 & 0 & 1 & 2 \\ 0 & 1 & -2 & 4 & 3 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$ b. $\mathbf{B}=\begin{bmatrix} 2 & -1 & 2 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

c. $\mathbf{C}=\begin{bmatrix} 2 & -3 & 2 & 6 & -1 & 2 & -2 \\ 0 & 4 & 7 & 1 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 2 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

d. $\mathbf{D}=\begin{bmatrix} 2 & -3 & 2 & 6 & -1 & 2 & -2 \\ 0 & 4 & 7 & 1 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 2 & -3 & 2 \\ 0 & 0 & 0 & 0 & 5 & 1 & 3 \end{bmatrix}$

e. $\mathbf{E}=\begin{bmatrix} 2 & -3 & 2 & 6 & -1 & 2 & -2 \\ 0 & 4 & 7 & 1 & 1 & 5 & 0 \\ 0 & 2 & 0 & 0 & 2 & -3 & 2 \\ 0 & 0 & 0 & 0 & 5 & 1 & 3 \end{bmatrix}$

f. $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Çözüm:

9. $\mathbf{A}=\begin{bmatrix}4&2&1&3\\6&3&4&7\\2&1&0&1\end{bmatrix}$ matrisinin rankını bulunuz.

Çözüm: Matrisin boyutu 3×4 olduğundan rank en fazla 3 olabilir. **A** matrisinin 3×3 boyutlu tüm alt matrisleri;

$$\mathbf{A}_1=\begin{bmatrix}4&2&1\\6&3&4\\2&1&0\end{bmatrix}\quad \mathbf{A}_2=\begin{bmatrix}4&2&3\\6&3&7\\2&1&1\end{bmatrix}$$

$$\mathbf{A}_3=\begin{bmatrix}4&1&3\\6&4&7\\2&0&1\end{bmatrix}\quad \mathbf{A}_4=\begin{bmatrix}2&1&3\\3&4&7\\1&0&1\end{bmatrix}$$

Matrislerin determinantları;

$$|\mathbf{A}_1|=|\mathbf{A}_2|=|\mathbf{A}_3|=|\mathbf{A}_4|=0$$

olduğundan $r(\mathbf{A})<3$. **A** matrisinin 2×2 boyutlu alt matrisleri;

$\mathbf{A}_5 = \begin{bmatrix} 4 & 7 \\ 0 & 1 \end{bmatrix}$ ve $|\mathbf{A}_5| \neq 0$ olduğundan,

$r(\mathbf{A})=2$ olarak belirlenir.

10. $\mathbf{A}=\begin{bmatrix}4 & 3 & 1\\ 2 & 0 & 2\\ 1 & 3 & -2\end{bmatrix}$ matrisinin rankını bulunuz.

Çözüm: Elemanter satır sütun işlemleri ile

$$\mathbf{A}=\begin{bmatrix}4 & 3 & 1\\ 2 & 0 & 2\\ 1 & 3 & -2\end{bmatrix}\Rightarrow R_1 \leftrightarrow R_3$$

$$\mathbf{A}\sim\begin{bmatrix}1 & 3 & -2\\ 2 & 0 & 2\\ 4 & 3 & 1\end{bmatrix}\Rightarrow \begin{matrix}-2R_1+R_2\\ -4R_1+R_3\end{matrix}$$

$$\mathbf{A}\sim\begin{bmatrix}1 & 3 & -2\\ 0 & -6 & 6\\ 0 & -9 & 9\end{bmatrix}\Rightarrow \begin{matrix}(-1/6)R_2\\ (-1/9)R_3\end{matrix}$$

$$\mathbf{A}\sim\begin{bmatrix}1 & 3 & -2\\ 0 & 1 & -1\\ 0 & 1 & -1\end{bmatrix}\Rightarrow -R_2+R_3$$

$$\mathbf{A} \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{B}$$

Sonuç olarak **A**~**B** denk matrisi elde edilir. $|\mathbf{B}| = 0$ olduğundan $r(\mathbf{B})<3$ ve 2×2 boyutlu bir alt matrisi,

$$\mathbf{B}_1 = \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$$

için $|\mathbf{B}_1| \neq 0$ olduğundan $r(\mathbf{B})=r(\mathbf{A})=2$.