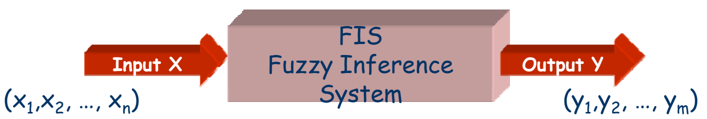
# Fuzzy Inference Systems

Lecture 08

# Fuzzy Inference Systems

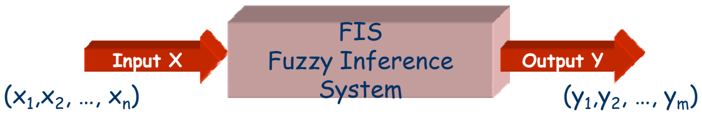



input singleton → output singleton

input fuzzy → output fuzzy

input crisp → fuzzification → fuzzy or singleton input → output

# Fuzzy Inference Systems

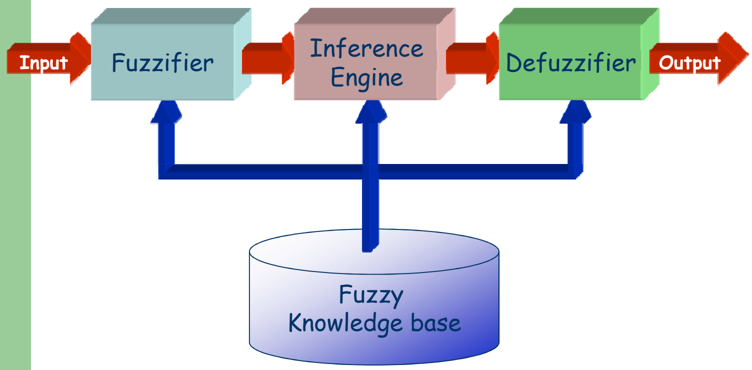



FIS has three main components:
1.) a rule base (fuzzy rules)
2.) a dictionary (MFs used in rules)
3.) a reasoning mechanism (inference procedure)

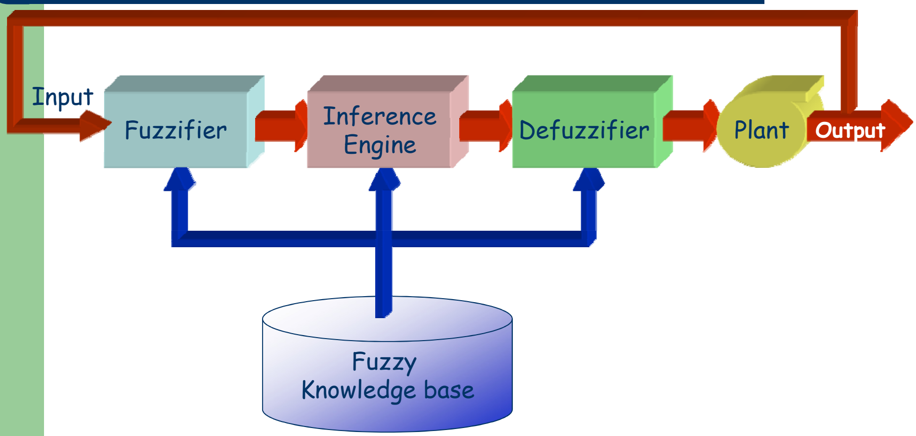
# Fuzzy Inference Systems

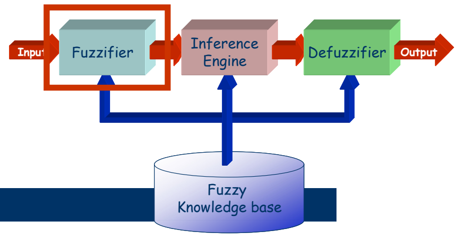
Fuzzy Inference system is also known by numerous other names:
- fuzzy rule based system
- fuzzy expert system
- fuzzy model
- fuzzy associative memory
- fuzzy logic controller
- fuzzy system

# Fuzzy Inference Systems



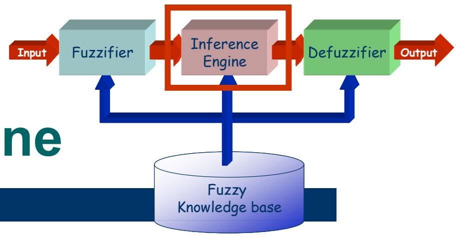

# Fuzzy Control Systems





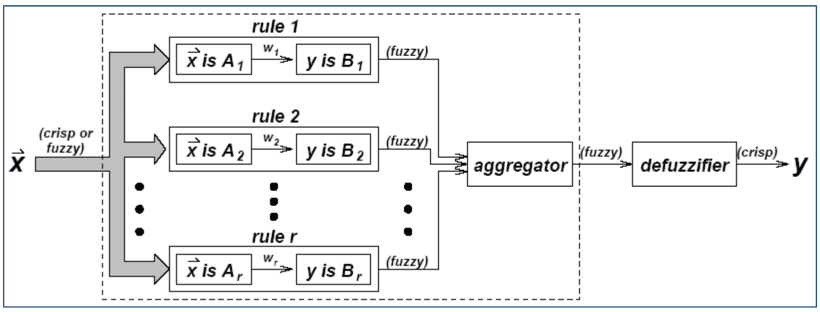

# Fuzzifier

Converts the crisp input to a linguistic variable using the membership functions stored in the fuzzy knowledge base.

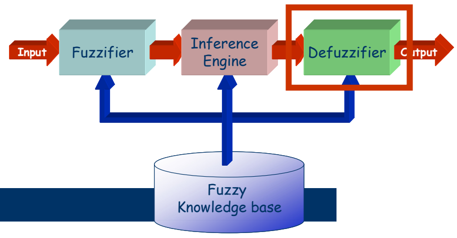



# Inference Engine

Using If-Then type fuzzy rules converts the fuzzy input to the fuzzy output.



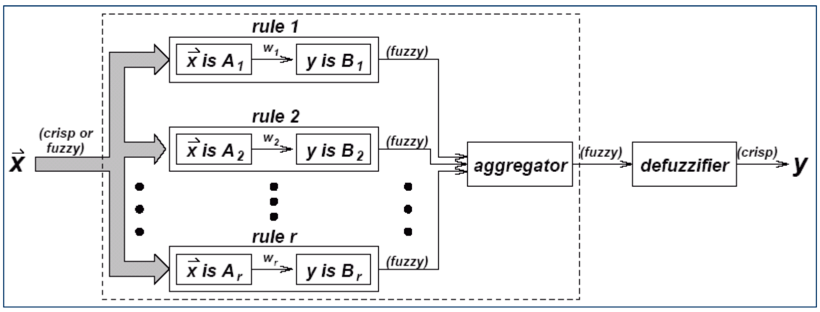
# Defuzzifier




Converts the fuzzy output of the inference engine to crisp using membership functions analogous to the ones used by the fuzzifier.

# Nonlinearity

A fuzzy inference system implements a nonlinear mapping from its input space to output space. This mapping is accomplished by a number of fuzzy IF-THEN rules each of which describes the local behavior of the mapping.



# FIS Types

There are different types of fuzzy inference systems. They differ from each other by the type of aggregators, basically. Also defuzzification methods may be different from each other.

- Mamdani Fuzzy models
- Sugeno Fuzzy Models
- Tsukamoto Fuzzy models

# Fuzzy Inference Systems

## Mamdani Fuzzy models

# Mamdani Fuzzy models

- Mamdani's fuzzy inference method is the most commonly seen fuzzy methodology. Mamdani's method was among the first control systems built using fuzzy set theory.
- It was proposed in 1975 by Ebrahim Mamdani [4] as an attempt to control a steam engine and boiler combination by synthesizing a set of linguistic control rules obtained from experienced human operators.
- Mamdani's effort was based on Lotfi Zadeh's 1973 paper on fuzzy algorithms for complex systems and decision processes [13]. Although the inference process described in the next few sections differs somewhat from the methods described in the original paper, the basic idea is much the same.

# Mamdani Fuzzy models

MAMDANI FUZZY INFERENCE
(Also known as max-min implication)

Input (s) : Fuzzy

Output(s): Fuzzy

Rule Base: connected via “AND”, “OR” or “ELSE”.

# Mamdani Fuzzy models

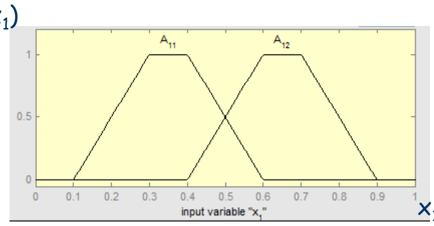
Example-1. A two-inputs one-output Mamdani model:

Suppose the input $X=\{x_1,x_2\}$ and the output $Y=\{y\}$ is given as in the following:

Input-1: $x_1$
(with two fuzzy trapMFs)

( $x_1 \in [0,1]$ )

$\mu(x_1)$



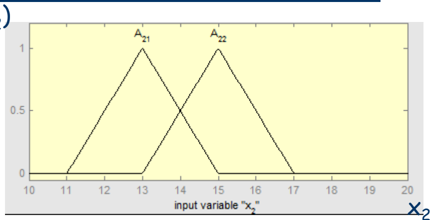

$x_1$

# Mamdani Fuzzy models

Input-2: $x_2$
(with two fuzzy triMFs)

( $x_2 \in [10,20]$ )

$\mu(x_2)$

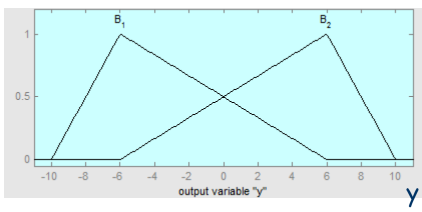



$x_2$

Output: y
(with two fuzzy triMFs)

( $y \in [-10,10]$ )

$\mu(y)$




y

# Mamdani Fuzzy models

Also, the fuzzy rules of the system are given as:

Fuzzy IF-THEN Rules of the system:

1. IF ($x_1$ is $A_{11}$) AND ($x_2$ is $A_{22}$) THEN ($y$ is $B_1$)
2. IF ($x_1$ is $A_{12}$) AND ($x_2$ is $A_{21}$) THEN ($y$ is $B_2$)

Then, Mamdani (max-min) aggregator is:

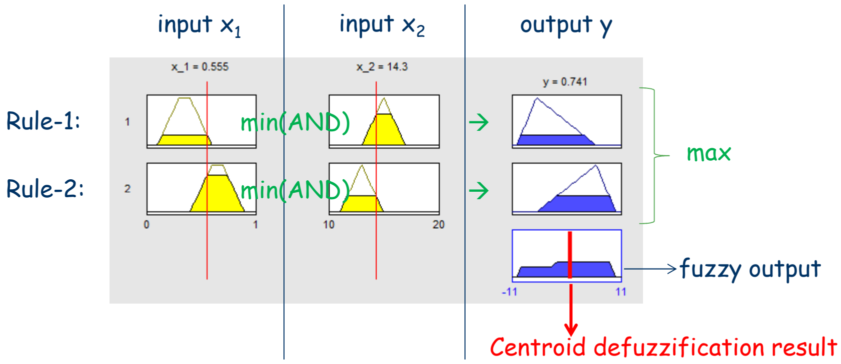
$$\mu_{B_k}(y) = \max_{\forall k}\left[\min\left(\mu_{A_{k(x_1)}}(\text{input } i), \mu_{A_{k(x_2)}}(\text{input } j)\right)\right]$$

$k = 1, 2, \ldots, r$

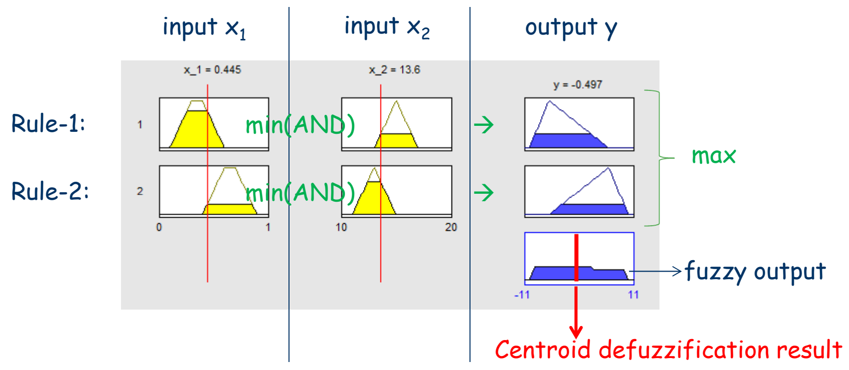
with $r$: number of total rules

Rules are connected with AND, so min operator is used. If rules were connected with OR, we would use max operator instead.

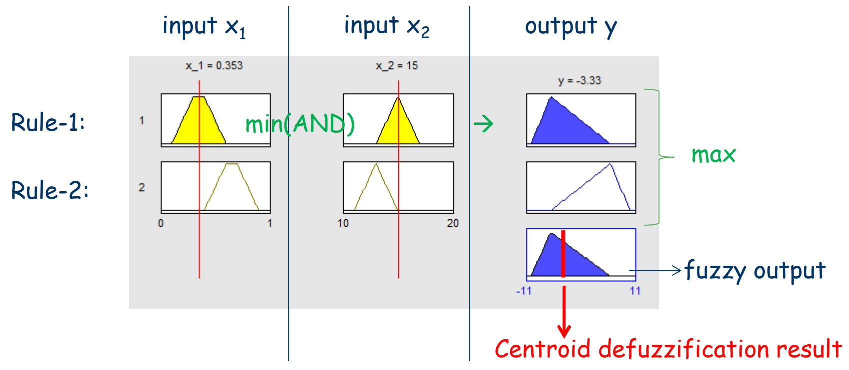
## Graphical Representation of Mamdani FIS



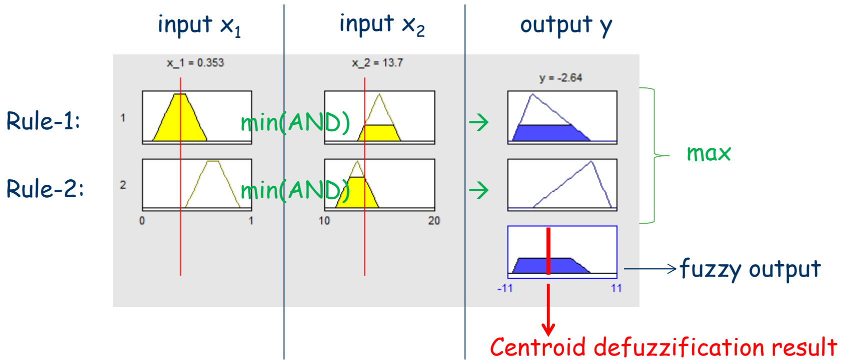
## Graphical Representation of Mamdani FIS



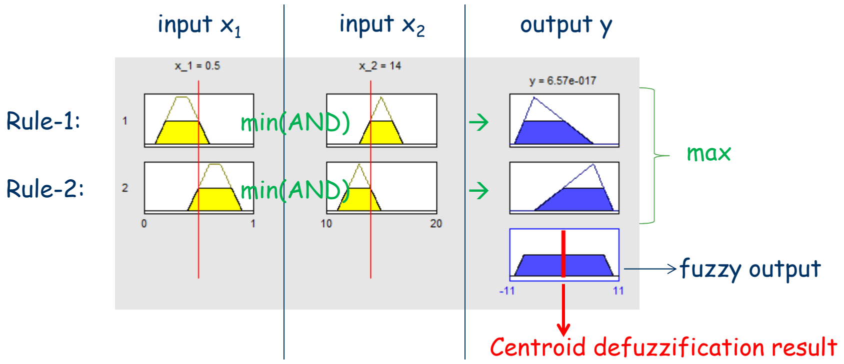

## Graphical Representation of Mamdani FIS



## Graphical Representation of Mamdani FIS




## Graphical Representation of Mamdani FIS



# The Reasoning Scheme of Mamdani Inference

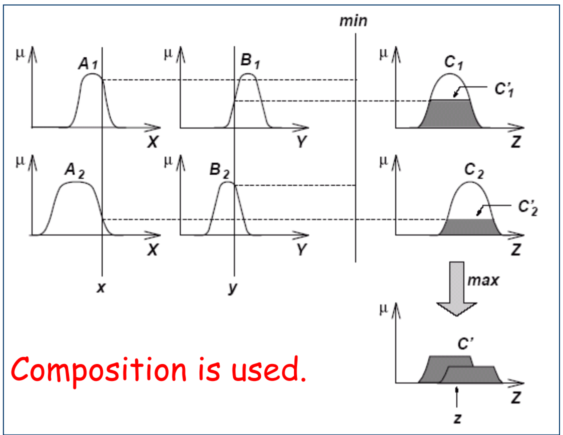
If rules are connected with AND:

-Max-Min composition

-Max-Product composition

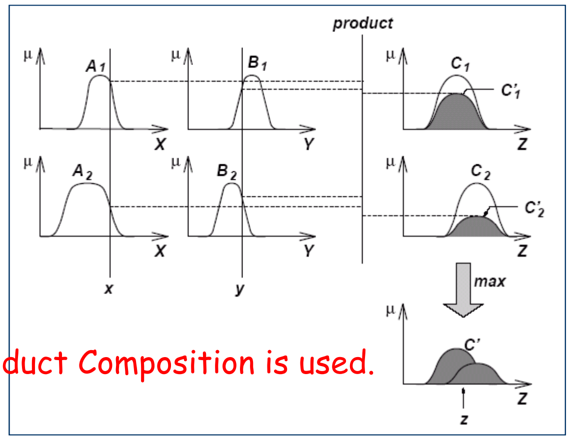
schemes can be used.

# The Reasoning Scheme



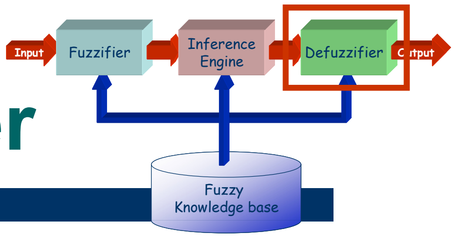

Max-Min Composition is used.

# The Reasoning Scheme



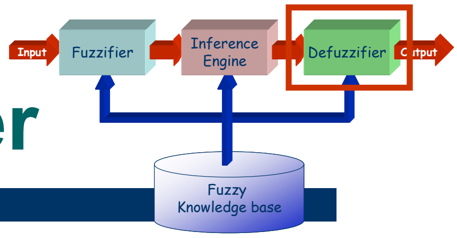

Max-Product Composition is used.

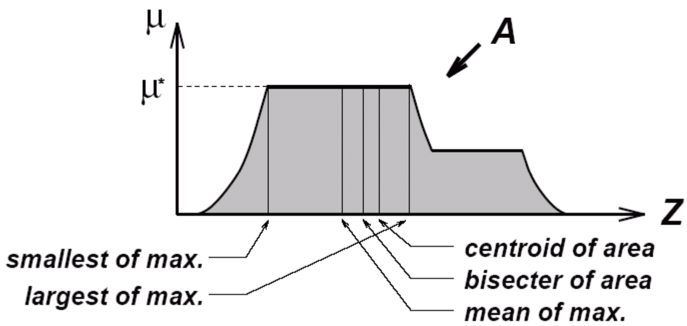
# Defuzzifier



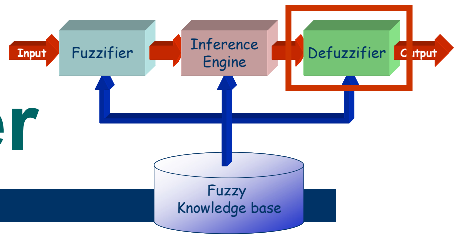

- Converts the fuzzy output of the inference engine to crisp using membership functions analogous to the ones used by the fuzzifier.
- Five commonly used defuzzifying methods:
  - Centroid of area (COA)
  - Bisector of area (BOA)
  - Mean of maximum (MOM)
  - Smallest of maximum (SOM)
  - Largest of maximum (LOM)

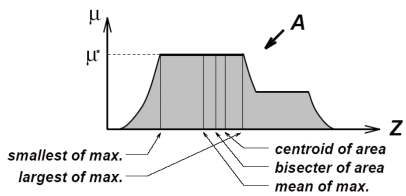
# Defuzzifier




# Defuzzifier




$$z_{COA} = \frac{\int_Z \mu_A(z) z dz}{\int_Z \mu_A(z) dz},$$

$$\int_{\alpha}^{z_{BOA}} \mu_A(z) dz = \int_{z_{BOA}}^{\beta} \mu_A(z) dz,$$

$$z_{MOM} = \frac{\int_{Z'} z dz}{\int_{Z'} dz},$$

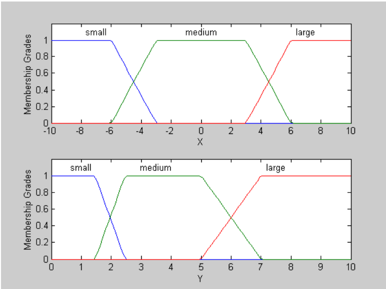
where $Z' = \{z; \mu_A(z) = \mu^*\}$

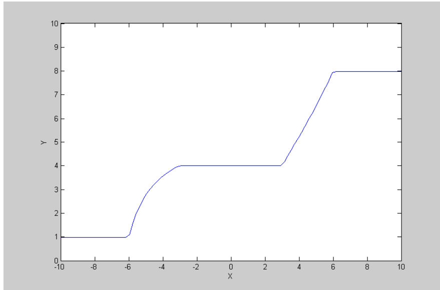
Rule1 : If $X$ is small then $Y$ is small
Rule2 : If $X$ is medium then $Y$ is medium
Rule3 : If $X$ is large then $Y$ is large

## Example-2: 1-input 1-output FIS:

$X$ = input $\in [-10, 10]$
$Y$ = output $\in [0, 10]$

Max-min composition and centroid defuzzification were used.



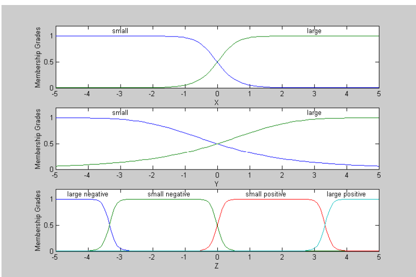

Overall input-output curve

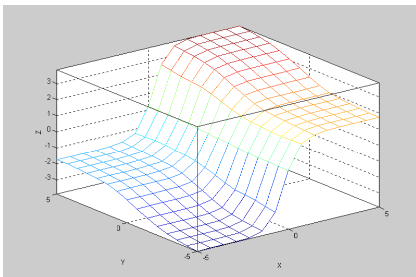
R1: If $X$ is small & $Y$ is small then $Z$ is negative large
R2: If $X$ is small & $Y$ is large then $Z$ is negative small
R3: If $X$ is large & $Y$ is small then $Z$ is positive small
R4: If $X$ is large & $Y$ is large then $Z$ is positive large

## Example-3

$X, Y, Z \in [-5, 5]$

Max-min composition and centroid defuzzification were used.




Overall input-output curve

# Fuzzy Inference Systems

Sugeno
Fuzzy Models

## Sugeno Fuzzy Models

- Also known as TSK fuzzy model
  - Takagi, Sugeno & Kang, 1985
- Goal: Generation of fuzzy rules from a given input-output data set.

# Fuzzy Rules of TSK Model

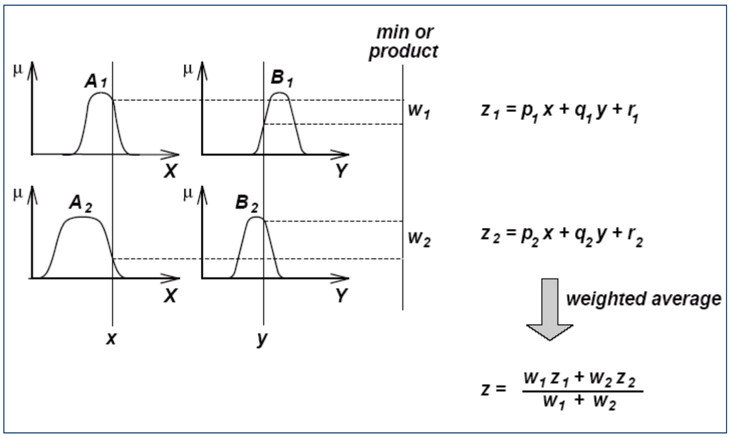
If $x$ is $A$ and $y$ is $B$ then $z = f(x, y)$

Fuzzy Sets

Crisp Function

$f(x, y)$ is very often a polynomial function w.r.t. $x$ and $y$.

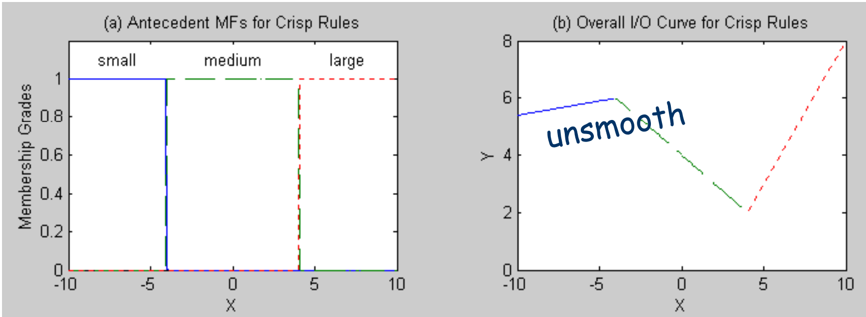
# The Reasoning Scheme



## Example-4:
## single-input single-output Sugeno model

R1: If $X$ is small then $Y = 0.1X + 6.4$
R2: If $X$ is medium then $Y = -0.5X + 4$
R3: If $X$ is large then $Y = X - 2$

$X$ = input $\in [-10, 10]$




R1: If $X$ is small then $Y = 0.1X + 6.4$
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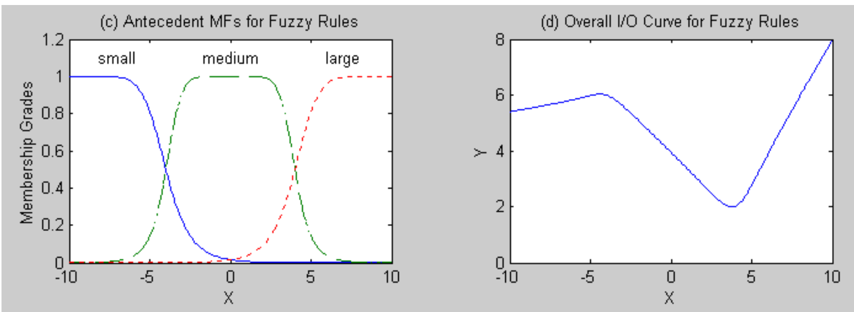
R3: If $X$ is large then $Y = X - 2$

$X$ = input $\in [-10, 10]$




If we have smooth membership functions (fuzzy rules) the overall input-output curve becomes a smoother one.

## Example-5:
A 2-input 1-output Sugeno model

R1: if $X$ is small and $Y$ is small then $z = -x + y + 1$

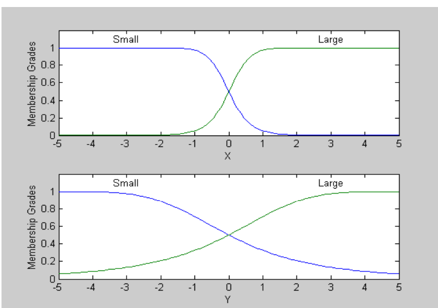
R2: if $X$ is small and $Y$ is large then $z = -y + 3$

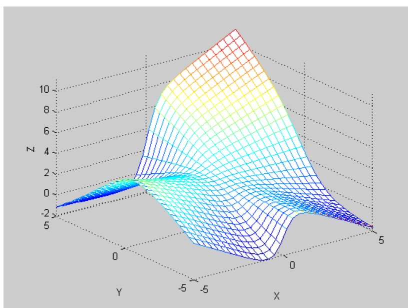
R3: if $X$ is large and $Y$ is small then $z = -x + 3$

R4: if $X$ is large and $Y$ is large then $z = x + y + 2$

R1: if $X$ is small and $Y$ is small then $z = -x + y + 1$
R2: if $X$ is small and $Y$ is large then $z = -y + 3$
R3: if $X$ is large and $Y$ is small then $z = -x + 3$
R4: if $X$ is large and $Y$ is large then $z = x + y + 2$

$X, Y \in [-5, 5]$




Output: A 3-D surface plot $z = f(x,y)$

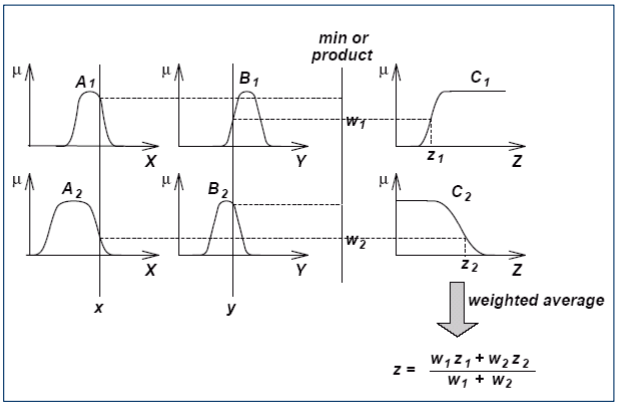
# Fuzzy Inference Systems

Tsukamoto
Fuzzy models

## Tsukamoto Fuzzy models

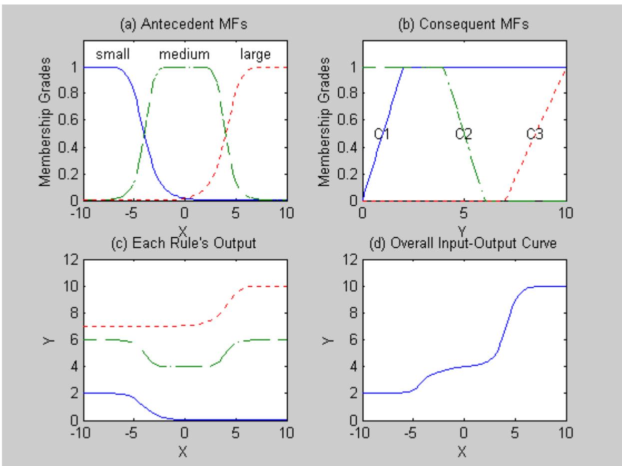
The consequent of each fuzzy if-then-rule is represented by a fuzzy set with a monotonical MF.

# Tsukamoto Fuzzy models




Example 6:

R1: If $X$ is small then $Y$ is $C_1$
R2: If $X$ is medium then $Y$ is $C_2$
R3: if $X$ is large then $Y$ is $C_3$



# Review of Fuzzy Models

If <antecedence> then <consequence>.

The same style for (<antecedence>)
- Mamdani Fuzzy models
- Sugeno Fuzzy Models
- Tsukamoto Fuzzy models

Different styles for (<consequence>)
- Mamdani Fuzzy models
- Sugeno Fuzzy Models
- Tsukamoto Fuzzy models

# Sugeno vs. Mamdani fuzzy systems

Advantages of the Mamdani Method

- It is intuitive.
- It has widespread acceptance.
- It is well suited to human input.

Advantages of the Sugeno Method

- It is computationally efficient.
- It can be used to model any inference system in which the output membership functions are either linear or constant.
- It works well with linear techniques (e.g., PID control).
- It works well with optimization and adaptive techniques.
- It has guaranteed continuity of the output surface.
- It is well suited to mathematical analysis.

## Example-7: Tipper example

We want to determine the tipper amount after eating a dinner in a restaurant. We have 2 criteria (inputs):

- Service quality: (grade between 0-10 points) (poor-good-excellent)
- Food quality: (grade between 0-10 points) (rancid-delicious)

We think that an average tip is 15%; a generous tip is 25% and a cheap tip is 5% of total amount of the bill.

## Example-7: Tipper example

Also, according to us, the following IF-THEN rules are suitable.

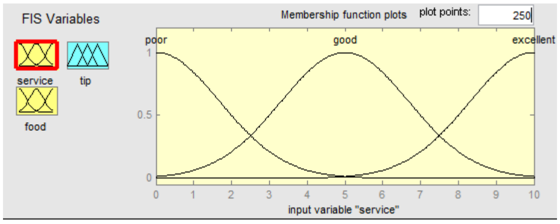
Rule-1: IF (service is poor) OR (food is rancid) THEN (tip is cheap)

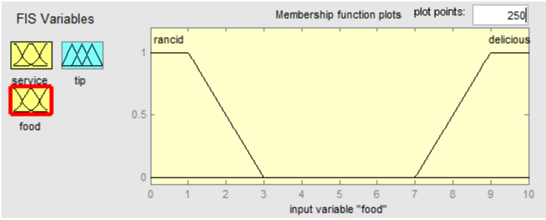
Rule-2: IF (service is good) THEN (tip is average)

Rule-3: IF (service is excellent) OR (food is delicious) THEN (tip is generous)

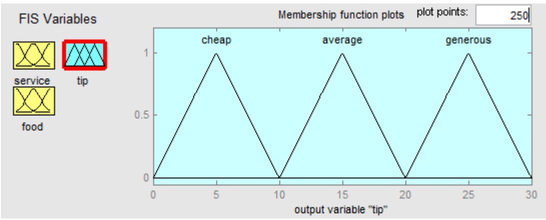
Let's design a MAMDANI FIS to determine the tipper amount.

## Example-7: Tipper example



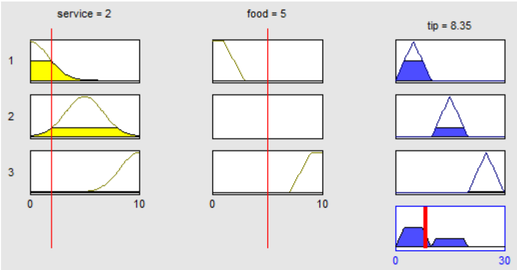

## Example-7: Tipper example




$$\mu_{tip}(y) = \max_{\forall k}\left[ \max\left( \mu_{service}(\text{input } i), \mu_{food}(\text{input } j) \right) \right]$$

Rules are connected with OR, so we need to take MAX.

## Example-7: Tipper example




Service = 2 pts, Food = 5 pts → tip = 8.35%

## Example-7: Tipper example


Service = 2 pts, Food = 10 pts → tip = 16.2%

## Example-7: Tipper example


Service = 7 pts, Food = 8 pts → tip = 20%