Fuzzy Extension Principle and Fuzzy Arithmetic

Lecture 06

Extension Principle for Crisp Sets

\[ x \rightarrow f(x) \rightarrow y \]

Extension is mapping from \( x \)-to-\( y \) through \( f( \cdot ) \).

\( y = f(x) \)

\( f: X \rightarrow Y \) (From one universe to another universe)

\( x \): input variable

\( y \): input image of \( x \) under \( f(\text{output}) \)

\( y = f(x) \) \( \Rightarrow \) \( x = f^{-1}(y) \) (original image of \( y \))
Crisp Mapping

A mapping can also be expressed by a relation $R$ on the Cartesian space $X \times Y$ with the characteristic function:

$$\chi_R(x, y) = \begin{cases} 
1, & y = f(x) \\
0, & y \neq f(x)
\end{cases}$$
**Extension Principle for Crisp Sets**

Let $A$ be a crisp set defined on $X$. The mapping $y = f(x)$ will result in a set $B$ defined on $Y$ such that

$$B = f(A) = \{y \mid \forall x \in A, \ y = f(x)\},$$

and the characteristic function of $B$ will be

$$\chi_B(y) = \bigvee_{y = f(x)} \chi_A(x).$$

Here, $B$ is another crisp set.

**Example:** Let $X = \{-2, -1, 0, 1, 2\}$ and $A = \{0, 1\}$ defined on $X$. $y = |4x| + 2$ mapping is applied to $A$, find $B$.

- $x = -2 \Rightarrow y = 10$
- $x = -1 \Rightarrow y = 6$
- $x = 0 \Rightarrow y = 2$
- $x = 1 \Rightarrow y = 6$
- $x = 2 \Rightarrow y = 10$

Thus, $Y = \{2, 6, 10\}$. 
Extension Principle for Crisp Sets

Method #1:
Directly applying the formula:

\[ \chi_B(y) = \frac{V}{y=f(x)} \chi_A(x) \]

\[ \chi_A(0) = 1, \quad \chi_A(1) = 1 \]

\[ \chi_A(-2) = \chi_A(-1) = \chi_A(2) = 0 \]
Extension Principle for Crisp Sets

Method #2:
Use relation matrix

\[
\begin{bmatrix}
2 & 6 & 10 \\
-2 & 0 & 0 & 1 \\
-1 & 0 & 1 & 0 \\
R = 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
2 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-2 & -1 & 0 & 1 & 2 \\
A = [0 & 0 & 1 & 1 & 0] \\
\end{bmatrix}
\]

Then, \( B = A \circ R \)

\[
B = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \quad B = \{2, 6\}
\]

Fuzzy Mapping
Extension Principle for Fuzzy Sets

\( A \): fuzzy set defined on \( X \)
\( y = f(x) \): functional transform or mapping
\( B \): image of \( A \) on \( X \) under \( f \).

\( B \) is a fuzzy set having universe of discourse \( Y \).

\[ B = f(A) \]

\[ \mu_B(y) = \bigvee_{f(x)=y} \mu_A(x) \]

Fuzzy Extension Principle

General definition: Suppose \( f \) is a mapping from an n-dimentional Cartesian product space \( X_1 \times X_2 \times \ldots \times X_n \) to a one dimentional universe \( Y \) such that \( y = f(x_1, x_2, \ldots, x_n) \) and suppose \( A_1, A_2, \ldots, A_n \) are n fuzzy sets in \( x_1, x_2, \ldots, x_n \) respectively. Then, the image of \( A_1, A_2, \ldots, A_n \) under \( f \) is given as:

\[ \mu(y) = \max_{f(x_1, x_2, \ldots, x_n)=y} \left\{ \min(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n)) \right\} \]

Zadeh's extension principle
Fuzzy Extension Principle

Example:

\[ A = \left\{ \frac{0.1}{-2} + \frac{0.4}{-1} + \frac{0.8}{0} + \frac{0.9}{1} + \frac{0.3}{2} \right\}, \quad f(x) = x^2 - 3 \]

\[ B = \left\{ \frac{0.8}{-3} + \frac{(0.4 V 0.9)}{-2} + \frac{(0.1 V 0.3)}{1} \right\} \]

\[ B = \left\{ \frac{0.8}{-3} + \frac{0.9}{-2} + \frac{0.3}{1} \right\} \]
Definition: A fuzzy vector is a vector containing fuzzy membership values.

\[
\vec{a} = \{a_1, \ldots, a_n\} = \left\{\mu_{A(a_i)} \right\} = \left\{\mu_{A_i} \right\}, \quad i = 1, \ldots, n
\]

\[
\vec{b} = \{b_1, \ldots, b_m\} = \left\{\mu_{B(b_j)} \right\} = \left\{\mu_{B_j} \right\}, \quad j = 1, \ldots, n
\]

\[
B = f(A) \quad \text{can be determined directly from} \quad B = A \circ R \quad \text{using vector form:}
\]

\[
B = a \circ R \quad \text{where} \quad R \quad \text{is an} \ n \times m \ \text{fuzzy relation matrix}
\]

Composition: \(\max - \min\)

\[
\hat{B}_{j} = \max_{i} \left(\min\left(a_{i}, r_{ij}\right)\right), \quad \hat{B}_{j} : j^{th} \text{element of the fuzzy image} \ \hat{B}.
\]
**Fuzzy Extension Principle**

**Definition:** If the input is a single element (a fuzzy singleton), the image of this singleton will be fuzzy and this case is termed as fuzzy transform. 

\[ B \subset f(x_i) \Rightarrow \mu_{B(x_i)} = r_{ij} \]

**Example:**

Let \( U = \{1, 2, 3\} \) and \( A = \{0.6, 0.9, 1\} \) defined on \( U \)

Mapping: \( v = f(u) = 2u - 1 \Rightarrow f(A) = ? \)

\( U = \{1, 2, 3\} \Rightarrow V = \{1, 3, 5\} \)

and \( f(A) = \{\frac{0.6}{1} + \frac{0.9}{3} + \frac{1}{5}\} \)
Mapping of more than one input variable:

Suppose \( U_1 \) and \( U_2 \) inputs are mapped to \( V \) through \( f(x_1, x_2) \). If the mapping is one-to-one, the same membership grades result but if the mapping is not one-to-one, maximum membership grades mapping to the same output variable is accepted.

\[
\mu_2(u_1, u_2) = \max \left[ \min \left( \mu_1(x_1), \mu_2(x_2) \right) \right]
\]

If \( f(x_1, x_2) = V \)

Fuzzy Arithmetic and Fuzzy Numbers:

Let \( I \) and \( J \) be two fuzzy numbers with \( I \) defined on \( X \) and \( J \) defined on \( Y \), and let the symbol \( * \) denote a general arithmetic operation.

\[
* = \{ +, -, \times, \div \}
\]

Fuzzy Extension Principle

An arithmetic operation between these two fuzzy numbers, denoted \( I * J \), is a mapping to another universe, say \( Z \), and accomplished by using the extension principle:

\[
\mu_{I*J}(z) = \bigvee_{x * y = z} (\mu_I(x) \land \mu_J(y))
\]

Example:

\[
A = \{z | \text{approximately 2} \} = \left\{ \frac{0.6}{1} + \frac{1}{2} + \frac{0.8}{3} \right\}
\]

\[
B = \{z | \text{approximately 6} \} = \left\{ \frac{0.8}{5} + \frac{1}{6} + \frac{0.7}{7} \right\}
\]
**Fuzzy Extension Principle**

Let's map the product of \( \frac{2}{3} \) and \( \frac{6}{7} \) to a fuzzy number \( \frac{12}{21} \) = "approximately 12"

Extension principle:

\[
2 \times 6 = \left( \frac{0.6}{1} + \frac{1}{2} + \frac{0.8}{3} \right) \times \left( \frac{0.8}{5} + \frac{1}{6} + \frac{0.7}{7} \right) \\
= \left\{ \frac{\min(0.6,0.8)}{5} + \frac{\min(0.6,1)}{6} + \ldots + \frac{\min(0.8,0.7)}{21} \right\} \\
= \left\{ \frac{0.6}{5} + \frac{0.6}{6} + \frac{0.6}{7} + \frac{0.8}{10} + \frac{0.7}{12} + \frac{0.8}{14} + \frac{0.8}{15} + \frac{0.7}{21} \right\}
\]

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**Fuzzy Extension Principle**

\( I = \left[ \begin{array}{c} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{array} \right] \) Let's find \( I+I \) using the extension principle:

\[
1+1 = \left( \frac{0.2}{0} + \frac{1}{1} + \frac{0.2}{2} \right) + \left( \frac{0.2}{0} + \frac{1}{1} + \frac{0.2}{2} \right) \\
= \left\{ \frac{\min(0.2,0.2)}{0} + \max\left( \frac{\min(0.2,0.1)}{1}, \frac{\min(1,0.2)}{1} \right) + \ldots + \frac{\min(0.2,0.2)}{4} \right\} \\
= \left\{ \frac{0.2}{0} + \frac{0.2}{1} + \frac{0.2}{2} + \frac{0.2}{3} + \frac{0.2}{4} \right\}
\]
Fuzzy Extension Principle

Example:

\[ I = \left\{ \frac{0.3}{1}, \frac{1}{2}, \frac{0.7}{3} \right\} \]
\[ J = \left\{ \frac{0.2}{4}, \frac{1}{5}, \frac{0.6}{6} \right\} \]
\[ I + J = \left\{ \frac{0.3}{1} + \frac{0.7}{3}, \frac{1}{2}, \frac{0.6}{6} \right\} \]
\[ = \left\{ \frac{\text{min}(0.3, 0.2)}{5}, \frac{\text{max}(\text{min}(0.3, 1), \text{min}(1, 1.2))}{6}, \frac{\text{max}(\text{min}(0.3, 0.6), \text{min}(1, 1), \text{min}(0.7, 0.2))}{7} \right\} \]
\[ = \left\{ \frac{\text{max}(\text{min}(0.6, \text{min}(0.7, 1)), \text{min}(0.7, 0.6))}{8}, \frac{\text{min}(0.7, 0.6)}{9} \right\} = \left\{ \frac{0.2}{5}, \frac{0.3}{6}, \frac{1}{7}, \frac{0.7}{8}, \frac{0.6}{9} \right\} \]

Fuzzy Extension Principle

Example: We have two fuzzy sets \( \Lambda \) and \( B \), each defined on its own universe as follows:

\[ \Lambda = \left\{ \frac{0.2}{1}, \frac{1}{2}, \frac{0.7}{4} \right\} \quad \text{and} \quad B = \left\{ \frac{0.5}{1}, \frac{1}{2} \right\} \]

We wish to determine the membership values for the algebraic product mapping

\[ f(\Lambda, B) = \Lambda \times B \text{ (arithmetic product)} \]
\[ = \left\{ \frac{\text{min}(0.2, 0.5)}{1}, \frac{\text{max}[\text{min}(0.2, 1), \text{min}(0.5, 1)]}{2}, \frac{\text{max}[\text{min}(0.7, 0.5), \text{min}(1, 1)]}{4}, \frac{\text{min}(0.7, 1)}{8} \right\} \]
\[ = \left\{ \frac{0.2}{1}, \frac{0.5}{2}, \frac{1}{4}, \frac{0.7}{8} \right\} \]