Classical Relations & Fuzzy Relations

Lecture 03

Classical & Fuzzy Relations

Relation: Involved in logic and represents mapping

Crisp Relations: "completely, related " or "not related"

Fuzzy Relations: Allows the relations between elements of two or more sets.

Cartesian Product

Cartesian Product:

An orders sequence of r elements, written in the form $(a_1, a_2, \ldots, a_r)$ is called an ordered r-tuple.

Cartesian Product

For crisp sets $A_1, A_2, \ldots A_r$, the set of all r-tuples $(a_1, a_2, \ldots, a_r)$ where $a_1 \in A_1$, $a_2 \in A_2$, …, $a_r \in A_r$ is called the Cartesian Product of $A_1, A_2, \ldots, A_r$ and ia denoted by $A_1 \times A_2 \times \ldots \times A_r$.

Cartesian Product

Example: Two sets; $A=\{0,1\}$ and $B=\{a,b,c\}$.

$AxB = \{(0,a),(0,b),(0,c),(1,a),(1,b),(1,c)\}$

$BxA = \{(a,0),(a,1),(b,0),(b,1),(c,0),(c,1)\}$

$AxA = A^2 = \{(0,0),(0,1),(1,0),(1,1)\}$

$BxB = B^2 =$
$(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}$

Classical Relations

Crisp Relations:

A subset of the Cartesian Product $A_1 \times A_2 \times \ldots \times A_r$, is called on r-ary relation over $A_1, A_2, \ldots A_r$. If $r = 2$; $A_1 x A_2$ is called a binary relation from A_1 to A_2.

$$X \times Y(x,y) = \{(x,y) \mid x \in X, y \in Y\}$$

Classical Relations

The strength of this relationship between ordered pairs of elements in each universe is measured by the characteristic function, denoted, χ

Classical Relations

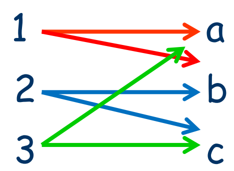
Characteristic Function:

$$\chi_{X \times Y}(x,y) = \begin{cases} 1, & (x,y) \in X \times Y \\ 0, & (x,y) \notin X \times Y \end{cases}$$

Finite discrete sets are related via relation matrix.

Classical Relations

Example:



$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \quad \text{or} \quad R = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Classical Relations

Definition:

Identity Relation: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$

Null Relation: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{zeros}$

Classical Relations

Complete Relation: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ = ones

Classical Relations

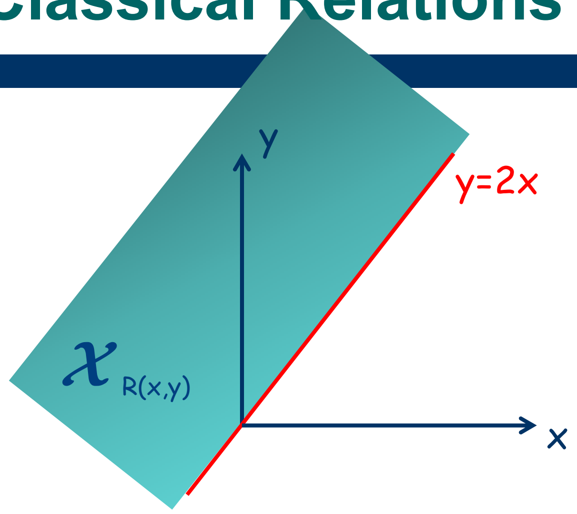
Example: Relations can also be defined for continuous universes.

Consider, for example,

R = {(x,y) | y≥2x, x∈X, y∈Y}. Then this means

$$\mathcal{X}_{R(x,y)} = \begin{cases} 1, & (x,y) \in X \times Y \\ 0, & (x,y) \notin X \times Y \end{cases}$$

Classical Relations



Classical Relations

Cardinality of Crisp Relations:

The cardinality of a set is a measure of the number of the elements of the set.

If cardinality of X is n_X and cordinality of Y is n_Y, then cordinality of the relation

$$n_{X \times Y} = n_X \times n_Y.$$

Classical Relations

Operations of Crisp Relations:

Union

$R \cup S \rightarrow \mathcal{X}_{R \cup S}(x,y) = \max[\mathcal{X}_R(x,y), \mathcal{X}_S(x,y)]$

Intersection

$R \cap S \rightarrow \mathcal{X}_{R \cap S}(x,y) = \min[\mathcal{X}_R(x,y), \mathcal{X}_S(x,y)]$

Classical Relations

Complement

$\bar{R} \rightarrow \mathcal{X}_{\bar{R}}(x,y) = 1 - \mathcal{X}_R\,(x,y)$

The properties of *commutativity*, *associativity*, *distributivity*, *involution* and *idempotency* all hold for crisp relations just as they do for classical set operations.

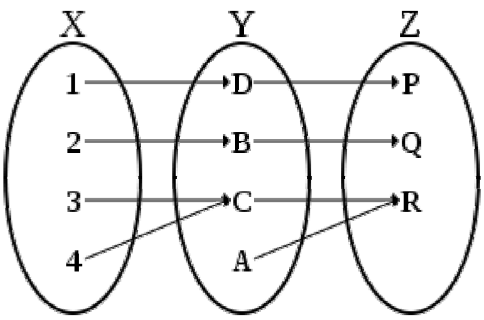
Classical Relations

Compositions:

Let R: X→Y and S: Y→Z.

We can find T: X→Z and denoted by T=R∘S. This is called as composition.

Composition



Some forms of composition operator

1. Max-Min Composition:

$$T = R \circ S \Rightarrow \mathcal{X}_{T(x,z)} = \bigvee_{y \in Y} (\mathcal{X}_R(x,y) \wedge \mathcal{X}_S(y,z))$$

2. Max-Product Composition:

$$T = R \circ S \Rightarrow \mathcal{X}_{T(x,z)} = \bigvee_{y \in Y} (\mathcal{X}_R(x,y) \cdot \mathcal{X}_S(y,z))$$

Classical Relations

Example:

T=R∘S using max-min composition?

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad S = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

Classical Relations

- Using the max–min composition operation, relation matrices for R and S would be expressed as

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{matrix} y_1 & y_2 & y_3 & y_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad S = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{matrix} z_1 & z_2 \end{matrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

$\mu_T(x1, z1) = \max[\min(1, 0), \min(0, 0), \min(1, 0), \min(0, 0)] = 0$

$$T = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{matrix} z_1 & z_2 \end{matrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

Classical Relations

- Using the max–min composition operation, relation matrices for R and S would be expressed as

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{matrix} y_1 & y_2 & y_3 & y_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad S = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \begin{array}{c} \begin{matrix} z_1 & z_2 \end{matrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

$\mu_T(x1, z1) = \max[\min(1, 0), \min(0, 0), \min(1, 0), \min(0, 0)] = 0$

$\mu_T(x1, z2) = \max[\min(1, 1), \min(0, 0), \min(1, 1), \min(0, 0)] = 1$

$$T = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{matrix} z_1 & z_2 \end{matrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

Classical Relations

R: X→Y, S:Y→Z ⇒ T:X→Z

Answer:

$$T = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

Fuzzy Relations

Fuzzy Relations:

A fuzzy relation $\underset{\sim}{R}$ is a mapping from Cartesian space X x Y to the interval [0,1]. The strength of mapping is expressed by the membership function of the relation for orders pairs from the two universes: $\mu_{\underset{\sim}{R}}(x,y)$.

Fuzzy Relations

Cardinality of Fuzzy Relations :

Since the cardinality of fuzzy sets on any universe is infinity, the cardinality of a fuzzy relation between two or more universes is also infinity.

$$\infty$$

Example (Approximate Equal)

$$X = Y = U = \{1, 2, 3, 4, 5\}$$

$$\underset{\sim}{R} = \left\{ \left((x, y), \mu_{\underset{\sim}{R}}(x, y) \right) \mid (x, y) \in X \times Y \right\}$$

$$\mu_{\underset{\sim}{R}}(u, v) = \begin{cases} 1 & |u - v| = 0 \\ 0.8 & |u - v| = 1 \\ 0.3 & |u - v| = 2 \\ 0 & otherwise \end{cases}$$

$$M_{\underset{\sim}{R}} = \begin{bmatrix} 1 & 0.8 & 0.3 & 0 & 0 \\ 0.8 & 1 & 0.8 & 0.3 & 0 \\ 0.3 & 0.8 & 1 & 0.8 & 0.3 \\ 0 & 0.3 & 0.8 & 1 & 0.8 \\ 0 & 0 & 0.3 & 0.8 & 1 \end{bmatrix}$$

Fuzzy Cartesian Product

Fuzzy Cartesian Product:

Let $\underset{\sim}{A}$ be a fuzzy set on X and $\underset{\sim}{B}$ be a fuzzy set on be a fuzzy set on Y,

$$\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{R} \subset X x Y,$$

$$\mu_{\underset{\sim}{R}}(x,y) = \mu_{\underset{\sim}{A} \times \underset{\sim}{B}}(x,y) = \min(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y))$$

Fuzzy Cartesian Product

Example:

$$\underset{\sim}{A} = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} \qquad \underset{\sim}{B} = \frac{0.3}{y_1} + \frac{0.4}{y_2}$$

$$\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{cc} y_1 & y_2 \\ \left[\begin{matrix} 0.2 & 0.2 \\ 0.3 & 0.4 \\ 0.3 & 0.4 \end{matrix}\right] \end{array}$$

Fuzzy Relation Operations

Let $\underset{\sim}{R}$ and $\underset{\sim}{S}$ be fuzzy relations on the Cartesian space X×Y,

Union : $\mu_{\underset{\sim}{R} \cup \underset{\sim}{S}}(x,y) = \max(\mu_{\underset{\sim}{R}}(x,y), \mu_{\underset{\sim}{S}}(x,y))$

Intersection: $\mu_{\underset{\sim}{R} \cap \underset{\sim}{S}}(x,y) = \min(\mu_{\underset{\sim}{R}}(x,y), \mu_{\underset{\sim}{S}}(x,y))$

Complement : $\mu_{\overline{\underset{\sim}{R}}}(x,y) = 1 - \mu_{\underset{\sim}{R}}(x,y)$

Classical & Fuzzy Relations

Commutativity, associativity, distributivity, involution and *idempotency* properties all hold for fuzzy relations. But **excluded middle laws do not hold**:

$\underset{\sim}{R} \cup \overline{\underset{\sim}{R}} \neq E$ E: Complete relation

$\underset{\sim}{R} \cap \overline{\underset{\sim}{R}} \neq 0$ 0: Null relation

Fuzzy Compositions

Fuzzy Compositions:

Fuzzy compositions can be defined just as it is for crisp (binary) relations (by using fuzzy sets instead of crisp sets).

$$\underset{\sim}{R} = X \times Y,\ \underset{\sim}{S} = Y \times Z \Rightarrow \underset{\sim}{T} = \underset{\sim}{R} \circ \underset{\sim}{S},\quad \underset{\sim}{T}: X \times Z.$$

Fuzzy Compositions

Max-Min Composition:

$$\mu_{\underset{\sim}{T}}(x,z) = \bigvee_{y \in Y} (\mu_{\underset{\sim}{R}}(x,y) \wedge \mu_{\underset{\sim}{B}}(y,z))$$

Max-Product Composition:

$$\mu_{\underset{\sim}{T}}(x,z) = \bigvee_{y \in Y} (\mu_{\underset{\sim}{R}}(x,y) \cdot \mu_{\underset{\sim}{B}}(y,z))$$

Fuzzy Compositions

Neither crisp nor fuzzy compositions are commutative in general

$$\tilde{R} \circ \tilde{S} \neq \tilde{S} \circ \tilde{R}$$

Fuzzy Compositions

Example:

X ={x_1, x_2}, Y ={y_1, y_2}, Z={z_1, z_2, z_3}

$$\tilde{R} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{matrix} y_1 & y_2 \end{matrix} \\ \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \tilde{S} = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{matrix} z_1 & z_2 & z_3 \end{matrix} \\ \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

$$\tilde{T} = \tilde{R} \circ \tilde{S} \,?$$

Fuzzy Compositions

Answers:

$$\underset{\sim}{T} = \begin{array}{c} \\ x_1 \\ x_2 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix} \end{array}$$

(Max-Min Comp.)

$$\underset{\sim}{T} = \begin{array}{c} \\ x_1 \\ x_2 \end{array}\begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.63 & 0.42 & 0.25 \\ 0.72 & 0.48 & 0.2 \end{bmatrix} \end{array}$$

(Max-Product Comp.)

Relation properties

Relations can exhibit various useful properties, a few of which are discussed here. Relations can be used especially in graph theory. Consider the following figure. This figure describes a universe of three elements, which are labeled as the vertices of this graph, 1, 2, and 3, or in set notation, X = {1, 2, 3}. The useful properties we wish to discuss are reflexivity, symmetry, and transitivity.

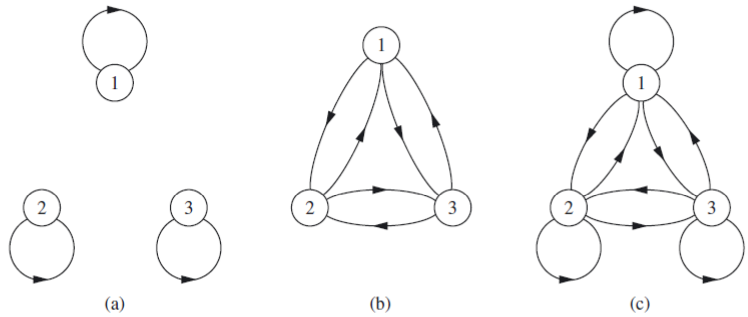


FIGURE 3.8
Three-vertex graphs for properties of (a) reflexivity, (b) symmetry, and (c) transitivity (Gill, 1976).

Relation properties

	CRISP	FUZZY
Reflexivity:	$(x_i, x_i) \in R$	$\mu_{\underset{\sim}{R}}(x_i, x_i) = 1$
Symmetry:	$(x_i, x_j) \in R \Rightarrow (x_i, x_j) \in R$	$\mu_{\underset{\sim}{R}}(x_i, x_j) = \mu_{\underset{\sim}{R}}(x_j, x_i)$

Relation properties

	CRISP	FUZZY
Transitivity:	$(x_i,x_j) \in R,\ (x_j,x_k) \in R$ $\Rightarrow (x_i,x_k) \in R$	$\mu\ (x_i,x_j) = \lambda_1,$ $\mu\ (x_j,x_k) = \lambda_2,$ $\Rightarrow \mu\ (x_i,x_k) = \lambda$ where $\lambda \geq \min(\lambda_1, \lambda_2)$

Special Relations

Equivalence Relations:

The relation R is an equivalence relation if it has the following three properties;

1) Reflexivity
2) Symmetry
3) Transitivity

Special Relations

Tolerance Relation:

A tolerance relation R on a universe X is a relation that exhibits only the properties of reflexivity and symmetry.

Classical & Fuzzy Relations

Example:

Suppose in an airline transportation system we have a universe composed of five elements: the cities Omaha, Chicago, Rome, London, and Detroit. The airline is studying locations of potential hubs in various countries and must consider air mileage between cities and takeoff and landing policies in the various countries.

Classical & Fuzzy Relations

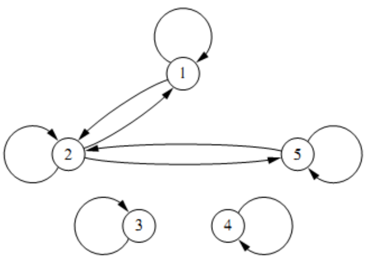
- These cities can be enumerated as the elements of a set, i.e.,
  $X = \{x_1, x_2, x_3, x_4, x_5\}$={Omaha, Chicago, Rome, London, Detroit}
- Suppose we have a tolerance relation, R_1, that expresses relationships among these cities:

$$R_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

This relation is reflexive and symmetric.

Classical & Fuzzy Relations

- The graph for this tolerance relation



- If $(x_1, x_5) \in R_1$ can become an equivalence relation

Example:

The following fuzzy relation is reflexive and symmetric. However, it is not transitive.

$$\underset{\sim}{R}_1 = \begin{bmatrix} 1 & 0.8 & 0 & 0.1 & 0.2 \\ 0.8 & 1 & 0.4 & 0 & 0.9 \\ 0 & 0.4 & 1 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0.5 \\ 0.2 & 0.9 & 0 & 0.5 & 1 \end{bmatrix}$$

$\mu_R(x_1, x_2) = 0.8$, $\mu_R(x_2, x_5) = 0.9 \geq 0.8$

but

$\mu_R(x_1, x_5) = 0.2 \leq \min(0.8, 0.9)$

Homework

Homework:

See extra examples from the textbook.